

Randomized measurements for large-scale quantum experiments

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MIT Seminar

Quobly was launched in November 2022



Quantum Information Team @ Quobly

Measurements & Benchmarking







Tensor-Network Simulations & quantum algorithms R&D (open-source)



Postprocessing



Observables, Entanglement, Fidelity, XEB Fidelity, Tomography, etc



Large scale compilation, Tensor quantum programming, <u>Qubitization</u>

Architectures and quantum error correction Surface-17 code



Internal Simulator

OUIMB

OPU

-W

|w|





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Moderate scale randomized measurements
1) Motivations
2) Measurements of entanglement entropies
3) More advanced usage and "statistical tricks"

Large scale learning via randomized measurements



Randomized measurements for large-scale experiments

Randomized measurements





The randomized measurement toolbox A. Elben, S. T. Flammia, H.-Y. Huang, R. Kueng, J. Preskill, BV, P. Zoller,



P. Zoller (UIBK) A. Elben (Psi)



J. Preskill, R. Huang (Caltech) R. Kueng (Linz) J.I.C Cirac (MPQ B. Kraus (TUM)



M. Serbyn. M. Ljubotina (ISTA)



M. Votto, W. Lam (UGA)

V. Vitale (Pasgal)

A. Rath (IQM)



C. Roos, R. Blatt (UIBK) And many more...



Nature Physics Review (2023).





L. Piroli (Unibo)

Experiments



Xiao Mi (Google) P.Jurcevic (IBM)

ШF



CNIS







LARGE-SCALE QUANTUM EXPERIMENTS

Nowadays several experiments realize large-scale programmable quantum many-body systems (~100 qubits) [Arute, Nature '19; Bluvstein, Nature '22]





WHAT'S THE POINT OF LARGE-SCALE QUANTUM EXPERIMENTS

Quantum computing

Solves a classical problem or quantum problem via a quantum algorithm



We hope to realize a computation that is not (easily) accessible to classical computers: Quantum advantage

Quantum simulation

Emulates many-body physics via an artifical quantum material



Credit Manoj Joshi (IQOQI)

We want to obtain meaningful physical properties



Entanglement: A central concept in quantum computing & quantum simulation



A and B are entangled iff $|\psi\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle$ Example with two qubits. The Bell state $|\psi\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$ is entangled

Entanglement is quantified via Entropies

$$S_{\rm vN} = -\text{Tr}[\rho_A \log(\rho_A)] \text{ von Neumann Entropies}$$
$$S_{\alpha} = \frac{1}{1-\alpha} \log[\text{Tr}(\rho_A^{\alpha})] \text{ Rényi entropies}$$

Purity

$$\operatorname{Tr}(\rho_A^2)$$



Entanglement: A central concept in quantum computing & quantum simulation





Andersen et al, Nature 2025

- Most basic non-classical feature of quantum computers.
- All important quantum algorithms involve quantum operations that generate entanglement.
- Quantum algorithms with a low level of entanglement can be efficiently simulated with a classical computer (Eisert RMP 2010)
- Universal predictions for large-scale quantum computers (eg Nahum PRX 2017)



Entanglement: A central concept in quantum computing & quantum simulation





• Distinguish quantum phases in quantum simulation, spots quantum phase transitions (Eisert, RMP 2010)

• Describes many-body quantum dynamics, thermalization, disorder physics, etc



Measurements in quantum experiments



 Quantum measurements: the state |s⟩ is measured with probability (s|ρ|s⟩, in a certain measurement basis

We have access to observables of the type $O|s\rangle = O(s)|s\rangle$.

$$\langle O \rangle = \sum_{s} \langle s | \rho | s \rangle O(s)$$
 (7)

 Measurement of 4^N observables: I can realize state tomography, i.e measure ρ.

• Can I measure the purity $Tr(\rho^2)$ more directly?



Randomized measurements: A single data acquisition procedure



Repeat N_UxN_M times

- Randomized measurements: We measure $P_u(s) = \langle s | u \rho u^{\dagger} | s \rangle$, $u = u_1 \otimes \cdots \otimes u_N$.
- *u_i* chosen independently from the circular unitary ensemble (CUE)
- We extract quantities of interest from the statistics of P_u(s), over random unitary transformations.

For example, the purity formula (Elben, BV, et al, PRL 2018)

$$\mathrm{Tr}(\rho^2) = 2^N E_u \Big[\sum_{s,s'} (-2)^{-D(s,s')} P_u(s) P_u(s') \Big]$$



Original protocol: van Enk-Beenakker (PRL 2012)

• Consider a single qubit



• We evaluate the statistics of $P_u(s) = \langle s | u \rho u^{\dagger} | s \rangle = \sum_{m,n} u_{s,m} \rho_{m,n} u_{s,n}^*$

$$E[[P_u(s)]^2] = \sum_{m,n,m',n'} \rho_{m,n} \rho_{m',n'} E[u_{s,m} u_{s,n'}^* u_{s,m'} u_{s,n'}^*]$$
(2)

• Using Random Matrix Theory (2-design identities)

$$\overline{[P_u(s)]^2} = \frac{1}{6} \sum_{m,n,m',n'} \rho_{m,n} \rho_{m',n'} \left(\delta_{m,n} \delta_{m',n'} + \delta_{m,n'} \delta_{m',n} \right) = \frac{1 + \operatorname{Tr}(\rho^2)}{6}$$
(3)

First demonsration: Brydges et al, Science 2019



• A programmable quantum simulator



 Goal: Understand entanglement growth in a quantum system



First demonstration: Brydges et al, Science 2019

Demonstration of randomized measurements with the measurement of the purity





Recent use of RM: Andersen et al, Nature 2025





Today's menu

Moderate scale randomized measurements
1) Motivations
2) Measurements of entanglement entropies
3) More advanced usage and "statistical tricks"

Large scale learning via randomized measurements



Experimental Robust Shadow Estimation and measurement of the quantum Fisher information (Vitale et al, PRX Q 2024)



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Beyond the purity: Classical Shadows as the modern framework to postprocess randomized measurements

Data are processed 'robust classical shadows' (Chen et al, PRX 2021, improving the seminal Caltech paper: Nature Physics 2020)

$$\tilde{\rho}^{(r,b)} = \bigotimes_{j=1}^{N} \left(\frac{3}{2F_{b}[j] - 1} U_{j}^{(r)^{\dagger}} |s_{j}\rangle \langle s_{j}| U_{j}^{(r)} + \frac{F_{z}[j] - 2}{2F_{b}[j] - 1} \mathbf{1} \right),$$

where the calibration data of each batch *b* gives access to $F_b[j]$. Estimations of functions of ρ are built based on the relation $E[\tilde{\rho}^{(r,b)}] = \rho$

$$\operatorname{tr}[\rho O] \simeq \frac{1}{N_u} \sum_r \operatorname{tr}[\hat{\rho}^{(r)}O], \ \operatorname{tr}[\rho^2] \simeq \frac{1}{N_u(N_u-1)} \sum_{r_1 \neq r_2} \operatorname{tr}[\hat{\rho}^{(r_1)}\hat{\rho}^{(r_2)}]$$



(2)

Quantum Fisher information measurements (Vitale et al, PRX Q 2024)



Quantum Cramer Rao bound on parameter estimation

$$(\Delta heta)^2 \geq rac{1}{m F_{
m Q}[arrho,H]}, \qquad \qquad arrho(heta) = \exp(-i H heta) arrho_0 \exp(+i H heta),$$

The Quantum Fisher information (A=H, mixed state version, Braustein and Caves, PRL 1994)

$$F_Q = 2 \sum_{(i,j),\lambda_i+\lambda_j>0} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j} |\langle i|A|j\rangle|^2 \text{ with } \rho = \sum_i \lambda_i |i\rangle\langle i|$$



Mapping the quantity to a measurable "shadow' quantity

 Rath et al (PRL 2021): QFI is the limit of a series of MCO, i.e can be measured

$$F_n = 2 \operatorname{Tr} \left(\sum_{\ell=0}^n (\rho \otimes \mathbf{1} - \mathbf{1} \otimes \rho)^2 (\mathbf{1} \otimes \mathbf{1} - \rho \otimes \mathbf{1} - \mathbf{1} \otimes \rho)^\ell \mathbf{S}(A \otimes A) \right)$$
(10)

• The series convergences exponentially faster to the QFI: Example for noisy GHZ states (N = 10)



Multi-copy observable MCO

$$\operatorname{Tr}(O\rho^{\otimes n})$$



Quantum Fisher information

Statistical Errors

• The series of MCO *F_n* is estimated via randomized measurements

$$F_n = \operatorname{Tr}(O\rho^{\otimes n}) = E[\operatorname{Tr}(O\rho^{(r_1)} \otimes \cdots \otimes \rho^{(r_n)})]$$
(11)

• We provide analytical variance bounds ('error bars')

$$\mathbb{V}[\hat{F}_n] \leq \sum_{k=1}^n \frac{n!^2 2^{kN}}{k!(n-k)!^2 (M-k+1)^k} \operatorname{Tr}([O_k]^2)$$
$$O_k = \frac{1}{n!} \sum_{\pi} \operatorname{Tr}_{\{k+1\dots n\}} (\pi^{\dagger} O \pi [\mathbf{1}^{\otimes k} \otimes \rho^{\otimes (n-k)}]) \qquad (12)$$

There will be a tradeoff in choosing n to control both systematic and statistical errors

We will probably need many measurements and postprocessing time, we will need some tricks to improve on that..

Trick 1: Common randomized measurements (BV et al, PRX Quantum 2023)

In some experimements we have a reference state in mind (eg GHZ state)

Reference state

$$\tilde{\rho}_{\sigma}^{(r_i)} = \tilde{\rho}^{(r_i)} - \sigma^{(r_i)} + \overset{*}{\sigma}$$

$$\sigma^{(r)} = \sum_{\mathbf{s}} P_{\sigma}(\mathbf{s} | U^{(r_i)}) \bigotimes_{j=1}^{N} \left(3 U_j^{(r)\dagger} | s_j \rangle \langle s_j | U_j^{(r_i)} - 1 \right)$$

Robust classical Reference state shadow shadow

Such shifted shadow is still an unbiased estimator of the density matrix $\mathbb{E}[\hat{
ho}_{\sigma}^{(r_i)}] =
ho - \sigma + \sigma =
ho$

But with improved variance properties

$$\mathbb{V}[\hat{O}] \leq \frac{n^2 ||O_A^{(1)}||_2^2}{N_U} \left(3^{N_A} ||\rho_A - \sigma_A||_2^2 + \frac{2^{N_A}}{N_M} \right) + \mathcal{O}\left(\frac{1}{N_U^2}\right)$$

n-copy observable Functions of O and of the density matrix

Number of measurement settings

Number of shots per measurement settings **U** quobly

Trick 1: Batching (Rath et al PRX Quantum 2023)

We group several shadows into N_B>n batches to simplify postprocessing

$$\hat{\rho}_{\sigma}^{(b)} = \frac{N_B}{N_I} \sum_{i=(b-1)N_I/N_B+1}^{bN_I/N_B} \left(\sum_{r_i} \frac{\tilde{\rho}_{\sigma}^{(r_i)}}{N_U} \right)$$

U-statistics estimations (R. Huang et al, Nature Physics 2020)

$$\hat{F}_{0} = \frac{4(N_{B} - 2)!}{N_{B}!} \sum_{b_{1} \neq b_{2}} \operatorname{Tr}\left(\hat{\rho}_{\sigma}^{(b_{1})}[\hat{\rho}_{\sigma}^{(b_{2})}, A]A\right),$$
$$\hat{F}_{1} = 2\hat{F}_{0} - \frac{4(N_{B} - 3)!}{N_{B}!} \sum_{b_{1} \neq \dots \neq b_{3}} \operatorname{Tr}\left(\hat{\rho}_{\sigma}^{(b_{1})}\hat{\rho}_{\sigma}^{(b_{2})}[\hat{\rho}_{\sigma}^{(b_{3})}, A]A\right)$$

Postprocessing runtime: N_B^2 , N_B^3 , etc

Barely affects statistical errors in the `high accuracy regime'



Quantum Fisher information measurements (Vitale et al, PRX Q 2024)





Some recent uses of randomized measurements to access entanglement

Quantities/Concepts
Entropies
OTOCs
Spectral Form Factors
Mixed-state entanglement*
Cross-Platform verification*
Topological entropies
Symmetry resolved Entropies*
Entropies (live)
Operator entropies*
Quantum Mpemba effect
Quantum Fisher information (Robust):
Entropies (2D)
Entropies (Robust)

Entropies (large-scale) & tomography Superconducting qubits

lons Superconducting qubits lons

Platform

lons

lons

lons

Superconducting qubits

Superconducting qubits

Elben et al, PRL 2020 lons & superconducting qubits Elben et al, PRL 2020 Zhu et al, Nature Comm 2022 Superconducting qubits Satzinger al, Science 2021

Vitale et al, Sci Post 2022 lons Stricker et al, PRX Q 2022 lons

Rath et al, PRX Q 2023 Joshi et al, PRL 2024 Superconducting qubits Vitale et al, PRX Quantum 2024

Reference

Brydges et al Science 2019

Joshi et al, PRL 2020

Dong et al, PRL 2025

Andersen et al, Nature 2025

Hu et al, Nature Communications 16, 2943 (2025)

Votto et al, in preparation



RandomMeas.jl: Open-Source Package

RandomMeas: The randomized measurement toolbox in Julia



This package provides efficient routines for sampling, simulating, and post-processing randomized measurements, including classical shadows, to extract properties of many-body quantum states and processes. RandomMeas relies heavily on <u>ITensors.jl</u>.



https://github.com/bvermersch/RandomMeas.jl & Julia's General Registry

Classical shadows

- 1. Energy/Energy variance measurements with classical shadows
- 2. Robust Shadow tomography
- 3. Process Shadow tomography
- 4. Classical shadows with shallow circuits
- 5. Virtual distillation

A. Elben (Psi)

Quantum benchmark

- 6. Cross-Entropy/Self-Cross entropy benchmarking
- 7. Fidelities from common randomized measurements
- 8. Cross-Platform verification

Entanglement

- 9. Entanglement entropy of pure states"
- 10. Analyzing the experimental data of Brydges et al, Science 2019
- 11. Surface code and the measurement of the topological entanglement entropy
- 12. Mixed-state entanglement: The p3-PPT condition and batch shadows

Miscellanous

13. Noisy circuit simulations with tensor networks



Summary first part & transition

Many physical quantities have been recently measured with RM: Experimentally friendly acquisition, and numerically friendly "universal" postprocessing.

Robustness/Practical aspects well understood and experimentally demonstrated

For the purity and related quantities, number of measurements scales as 2^{aN} , a ≈ 1 , implies typically N <14

More info in our review **"The randomized measurement toolbox"** Elben, Flammia, Huang, Kueng, Preskill, BV, Zoller, Nat Rev Phys 2023

Let's expand this toolbox to large-scale systems reconstructions!



Large-scale entropies and tomography via randomized measurements

Probing many-body quantum states in large-scale experiments with randomized measurements, BV et al, Phys. Rev. X 2025

Learning mixed quantums in large-scale experiments (to appear on arxiv)





P. Zoller (UIBK) L. Piroli (Unibo)





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J.I.C Cirac (MPQ

Connections to MPS/MPO Gibbs State Tomography Litterature

Baumgratz et al, NJP 2013 Torlai et al, Nature Comm 2023

Anshu et al, Nature Physics 2021 (Review) Joshi et al, Nature 2023

...



M. Votto (UGA)



C. Lancien (UGA)



Goal a Tomography via randomized measurements



The learned state can be used to extract physical properties Conveniently (no more batching, etc) and for error mitigation, etc Matrix-Product-Operator (MPO) Represents the quantum state as a 1D compressed object with linear cost in storage and manipulation

$$\sigma = \sum_{\{s_j\}, \{s'_j\}} M^{(1)}_{s_1, s'_1} M^{(2)}_{s_2, s'_2} \dots M^{(N)}_{s_N, s'_N} \left| \{s_j\} \right\rangle \left\langle \{s'_j\} \right\rangle$$

describe output states of one-dimensional noisy/finite depth quantumcircuits, as well as one-dimensional thermal states relevant to quantum simulation

K. Noh, L. Jiang, and B. Fefferman, Quantum 4, 318 (2020).

S. Cheng, C. Cao, C. Zhang, Y. Liu, S.-Y. Hou, P. Xu, and B. Zeng, Phys. Rev. Res. **3**, 023005 (2021).

F. Verstraete, J. J. García-Ripoll, and J. I. Cirac, Phys. Rev. Lett. **93**, 207204 (2004).

T. Kuwahara, A. M. Alhambra, and A. Anshu, Phys. Rev. X 11, 011047 (2021).



Learning mixed quantums in large-scale experiments: Technical statements

We assume the existence of a finite correlation length in the MPO framework.

This implies the approximate factorization condition (Vermersch al Phys. Rev. X 2024)

$$\left| \operatorname{tr}[\rho_{ABC}^2]^{-1} \frac{\operatorname{tr}[\rho_{AB}^2] \operatorname{tr}[\rho_{BC}^2]}{\operatorname{tr}[\rho_B^2]} - 1 \right| \le \alpha e^{-|B|/\xi_{\rho}^{(2)}}$$

This assumption is satisfied in 1D quantum circuits, quantum Gibbs states (Capel et al, arxiv: 2024 in particular)



Learning mixed quantums in large-scale experiments: Technical statements

Votto et al (arxiv:..) : In this case, one can learn the state using the inner product as cost function

$$\mathcal{F}_{\rm GM}(\rho,\sigma) = \frac{\mathrm{tr}[\rho\sigma]}{\sqrt{\mathrm{tr}[\rho^2\sigma^2]}} \quad \text{With the MPO} \quad \sigma = \sum_{\{s_j\},\{s'_j\}} M^{(1)}_{s_1,s'_1} M^{(2)}_{s_2,s'_2} \dots M^{(N)}_{s_N,s'_N} \left|\{s_j\}\right\rangle \left\langle\{s'_j\}\right\rangle$$

With arbitrary small total error and polynomially many measurements

Proof idea: Gradient w.r.t local tensors can be set to zero using the rule

$$\begin{split} \mathrm{tr}_{I_j \setminus \{j\}} [\sigma_{I_j} \partial_{M^{(j)}} \sigma_{I_j}] &= \mathrm{tr}_{I_j \setminus \{j\}} [\rho_{I_j} \partial_{M^{(j)}} \sigma_{I_j}] \\ \uparrow & \uparrow \\ \\ \mathsf{Classically computable} & \mathsf{Finite support "observable", thus efficiently measurable} \end{split}$$

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Learning mixed quantums in large-scale experiments: Demonstration



 Experimental setup: superconducting quantum processor (IBM Brisbane QPU), N_{tot} = 96 qubits, depth D = 1,2 kicked Ising model

$$U_i^X = e^{-i\pi X_i/8}, \ U_i^{ZZ} = e^{i\pi Z_i Z_{i+1}/4}$$

• We perform $M = N_u \times N_M = D \cdot 3072 \times 1024$ measurements



Learning mixed quantums in large-scale experiments: Demonstration



Experimental tomography of a N = 96 mixed qubit state (previous results: N = 20 qubits, MPS (Kurmapu et al PRXQ 2023) or Gibbs states (Joshi et al, Nature 2023)

The MPO σ captures the noisy features of the experiment.



Learning mixed quantums in large-scale experiments: Extracting physical quantities

Pure state



(lenghty calculation)

 $D=2, N_{\mu}^{L}=3072, \ell=3$ Purity Entropy 20 10^{-6} 15 10⁻⁴ $\sim 10^{1}$ 10^{-2} 10^{0} C 32 80 48 96 64 16 N $-\log(tr[\sigma^2])$ k=3k = 4 $-\log(\hat{\mathcal{P}}_{2,AFC}^{(k=1)})$ k = 2 \bigcirc



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Learning mixed quantums in large-scale experiments: Applications

Quantum Error Mitigation:

Based on a noisy experimental device, estimation of noiseless observable estimation values based on noise extrapolation, noise models & extra sampling, virtual distillation, etc (Cai et al, RMP 2023)

$$|\psi\rangle = U |\psi_0\rangle \longrightarrow \rho = \mathcal{U}(\gamma)\rho_0 \longrightarrow \langle O \rangle = \langle \psi | O | \psi \rangle$$

Here: Quantum Principal Component analysis (stronger version of virtual distillation) via the DMRG algorithm of tensor-networks (Review: Schollwoeck, Annals of Physics, 2011)

$$\rho = \sum_{a} \Lambda_a |\psi_a^{\rho}\rangle \langle \psi_a^{\rho}|, \text{ where } \Lambda_a > \Lambda_{a+1}$$

$$DMRG-QPCA$$

$$\sigma |\psi_{\rm MPS}\rangle = \begin{array}{c} & & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\$$



We search classically for the ground state of $H = -\sigma$

Learning mixed quantums in large-scale experiments: Applications



Large-scale resconstruction of noiseless expectation values, and von Neumann entanglement entropies (Despite the original exponentially small original mixed state)



Conclusions and Thank you

The Quantum Information Team at Quobly



Benoît Vermersch



Thibaud Louvet

Carlos Ramos-Marimón



Dimitri Lanier



Amara Keita

Contact me if you want to know more, or join!

Probing many-body quantum states in large-scale experiments with randomized measurements, Phys. Rev. X 2025



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J.I.C Cirac (MPQ

Learning mixed quantums in large-scale experiments (to appear on arxiv)



M. Votto



(UGA)	C.



Quobly was born from the combined expertise of CEA and CNRS at Grenoble







Kraus

Joshi

$$\frac{d}{dt}\rho(t) = -i \ [H,\rho(t)] + D[\rho(t)]$$

$$H_{ij} = \sum_{\sigma_i,\sigma_j} h_{\sigma_i,\sigma_j} \sigma_i \sigma_j$$
$$D[\rho] = \sum_{i=1}^{N} D_i[\rho] \quad D_i[\rho] = \sum_{\sigma_i,\sigma'_i} d_{\sigma_i,\sigma'_i} \left(\sigma_i \rho \sigma'_i - \frac{1}{2} \{ \sigma'_i \sigma_i, \rho \} \right)$$

Goal: Learn the coefficients

- Wiebe et al. "Hamiltonian learning and certification using quantum resources." *PRL* (2014).
- Holzäpfel et al. "Scalable reconstruction of unitary processes and Hamiltonians." Phys Rev *A* (2015).
- Bairey, Arad, and Lindner. "Learning a local Hamiltonian from local measurements", PRL (2019).
- Evans, Harper, Flamia, "Scalable Bayesian Hamiltonian learning", arXiv:1912.07636 (2019)
- Li, Zou, and Hsieh. "Hamiltonian tomography via quantum quench", *PRL* (2020).
- Hu et al. "Ansatz-free Hamiltonian learning with Heisenberg-limited scaling." arXiv preprint arXiv:2502.11900 (2025).
- Olsacher et al. "Hamiltonian and Liouvillian learning in weakly-dissipative quantum many-body systems." *Quantum Science and Technology* (2025).
- Many more

One key application: Verified quantum simulation: Cai et al, arxiv:2311.14818, Trivedi et al, Nature Comm. 2025



Theory: Stilck França, Daniel et al « Efficient and robust estimation of many-qubit {Hamiltonians », Nature. Comm. 2024

A closed linear system of equations, pair-wise

$$\begin{aligned} \frac{d}{dt} \operatorname{tr}(\rho_{ij}(t))O_{ij}\Big|_{t=0} &= \\ -i\sum_{\sigma_i,\sigma_j} \operatorname{tr}\left([\sigma_i\sigma_j,\rho_i\rho_j]O_iO_j\right) h_{\sigma_i,\sigma_j} &\rho_{ij} = \rho_i \otimes \rho_j \bigotimes_{n \neq i,j} \frac{\mathbb{I}_{\sigma_i,\sigma_j}}{2} \\ &+ \sum_{\sigma_i,\sigma_i'} \operatorname{tr}\left(\sigma_i\rho_i\sigma_i'O_i - \frac{1}{2}\{\sigma_i'\sigma_i,\rho_i\}O_i\right) d_{\sigma_i,\sigma_i'} &O_{ij} = \sigma_i\sigma_j \\ &+ \sum_{\sigma_j,\sigma_j'} \operatorname{tr}\left(\sigma_j\rho_j\sigma_j'O_j - \frac{1}{2}\{\sigma_j'\sigma_j,\rho_j\}O_j\right) d_{\sigma_j,\sigma_j'} \end{aligned}$$

Unique Pseudo-inverse for enough initial states and measurement observables

Postprocessing for each pair independtly gives a N² runtime postprocessing algorithm





Where the sum runs over the N_c "compatible" settings r defined such as and such that the rotation v maps O_{ij} to the computational basis

The probability to have a compatible setting is independent of system sizes





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a) 51 ion H&L learning