Probing the entanglement structure of many-body quantum states via partial-transpose moments

Quantum measurements theory conference, Bad Honnef

Benoît Vermersch

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Outline

From pure to mixed-state entanglement in many-body quantum systems

Mixed-state entanglement from local randomized measurements

Revealing entanglement structure via partial-tranpose moments

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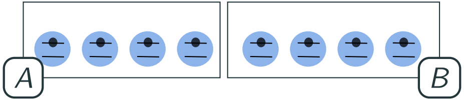
How to quantify pure-state entanglement?



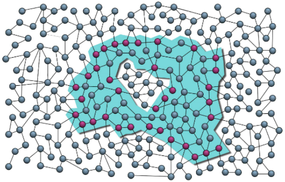
- For pure states, entropies of the reduced state $\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$ quantify entanglement

$$
\begin{aligned}
S &= -\mathrm{Tr}[\rho_A \log(\rho_A)] \text{ von Neumann entropies} \\
S_\alpha &= \frac{1}{1-\alpha} \log[\mathrm{Tr}(\rho_A^\alpha)] \text{ Rényi entropies}
\end{aligned}
$$

- Entanglement entropies quantity quantum resources and quantum information (*Excellent notes from J. Preskill on Quantum Shannon Theory*)

Entropies can distinguish the entanglement structure of many-body pure quantum states

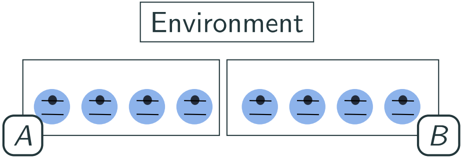
- Haar random states $|\psi\rangle = U|0\rangle^{\otimes N}$ with U *randomly* sampled from the Haar measure (CUE)
  - Good description of chaotic systems in condensed matter and black holes
  - Obey a volume law $S \propto \min(N_A, N_B)$.
- For the groundstate of a gapped local Hamiltonian $\rightarrow$ area law $S \propto N_A^{(\text{boundary})}$



- Scaling laws of S also reveal phase transitions, a topological phase, etc Eisert RMP 2010

Mixed-state entanglement

- Typically the state of a quantum system is 'mixed' due to an environment



- We define the density matrix of the quantum computer

$$\rho = \mathrm{Tr}_E(|\psi\rangle\langle\psi|)$$

- A and B are entangled iff

$$\rho \neq \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)}$$

- Entanglement entropies do not provide a complete description of mixed-state entanglement.

Mixed-state entanglement via the PPT criterion and negativity

- One approach (out of many): Peres-Horodecki PPT criterion:
- Consider the partial-transpose operation

$$\rho = \sum_{i_A, j_A, i_B, j_B} \rho_{i_A, j_A, i_B, j_B} |i_A, i_B\rangle \langle j_A, j_B| \rightarrow \rho^{\Gamma} = \sum_{i_A, j_A, i_B, j_B} \rho_{i_A, j_A, i_B, j_B} |j_A, i_B\rangle \langle i_A, j_B|$$

- If a state $\rho = \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)}$ is separable,

$$\rho^{\Gamma} = \sum_i p_i [\rho_A^{(i)}]^T \otimes \rho_B^{(i)} \text{ is positive semi-definite}$$

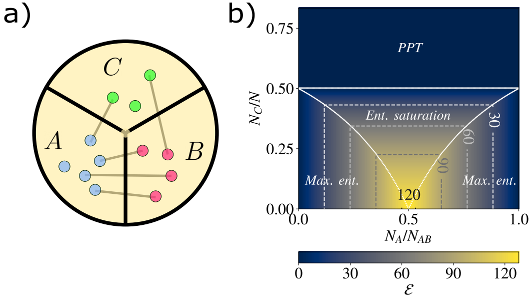
- The negativity $\mathcal{E} = \log \sum_{\lambda \in \mathrm{spec}(\rho^{\Gamma})} |\lambda|$ is an entanglement monotone (Vidal, 2001)

Negativity reveals the entanglement structure of mixed quantum states

- Consider a Haar random reduced mixed state ρ on $N = N_A + N_B + N_C$ qubits

$$\rho = \mathrm{Tr}_C(|\psi\rangle\langle\psi|)$$

- What is the average negativity as a function of the number of qubits N_A, N_B, N_C?
- Aubrun, Nechita, Shapourian, etc: There are three entanglement phases



- This talk: How to detect such phases in an experiment?

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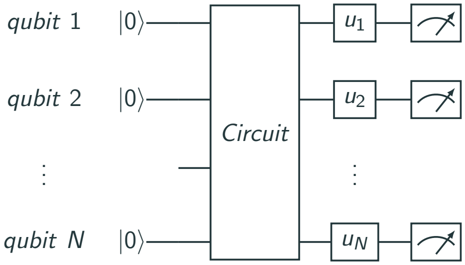
Mixed-state entanglement from local randomized measurements

- Joint work with A. Elben, R. Kueng, H-Y. Huang, R. van Bijnen, C. Kokail, M. Dalmonte, P. Calabrese, B. Kraus, J. Preskill, P. Zoller [Phys. Rev. Lett. 125, 200501 (2020)]
- Our starting point: The randomized measurement toolbox
- Goal: Detect entanglement via the PPT condition in a qubit system

The randomized measurements toolbox



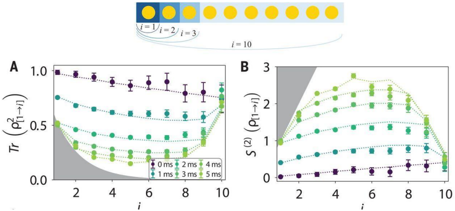
- Randomized measurements: We measure $P_u(s) = \langle s|u\rho u^\dagger|s\rangle$, $u = u_1 \otimes \cdots \otimes u_N$, $u_i \in \mathrm{CUE}$.
- We extract quantities of interest from the statistics of $P_u(s)$.

- Unbiased analytical estimators and performance guarantees
- Nonlinear functions of ρ require typically $\propto 2^N$ measurements
- See also impressive works by Andreas Ketterer, Lucas Knips, Otfried Gühne, et al

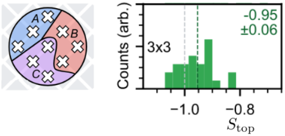
Review: The randomized measurement toolbox A. Elben, S. T. Flammia, H.-Y. Huang, R. Kueng, J. Preskill, B. V, P. Zoller, arXiv:2203.11374

Experimental uses of randomized measurements

Entanglement Rényi entropy growth, Brydges et al, Science 2019

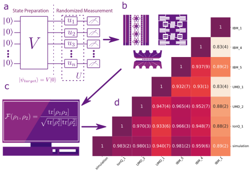


Topological entanglement entropy, Satzinger et al, Science 2021



Cross Platform Fidelities: Elben et al, PRL 2020,

Zhu et al, arXiv:2107.11387



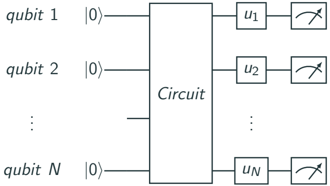
but also otocs, topological invariants, quantum Fisher information, etc

Measuring partial-transpose moments experimentally

- Consider the partial-transpose (PT) moments

$$p_n = \mathrm{Tr}[(\rho^{\Gamma})^n], \; n = 2, \ldots$$

- PT moments are experimentally accessible via local classical shadows (see also Gray, PRL 2017, Zhou PRL 2020)



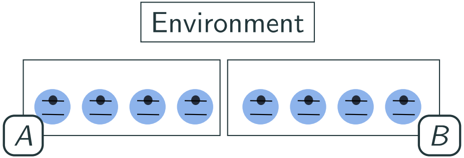
Classical shadows are built from bitstrings

$$\rho^{(r)} = \bigotimes_{i \in AB} \left(3u_i \left| k_i^{(r)} \right\rangle \left\langle k_i^{(r)} \right| u_i^{\dagger} - \mathbf{1}_i \right) \qquad E[\rho^{(r)}] = \rho$$

PT moments are extracted from bitstrings

$$\hat{p}_n = \frac{1}{n!} \binom{M}{n}^{-1} \sum_{r_1 \neq r_2 \neq \ldots \neq r_n} \mathrm{Tr}\left[(\hat{\rho}_{AB}^{(r_1)})^{\Gamma} \ldots (\hat{\rho}_{AB}^{(r_n)})^{\Gamma} \right].$$

The p_3-PPT condition

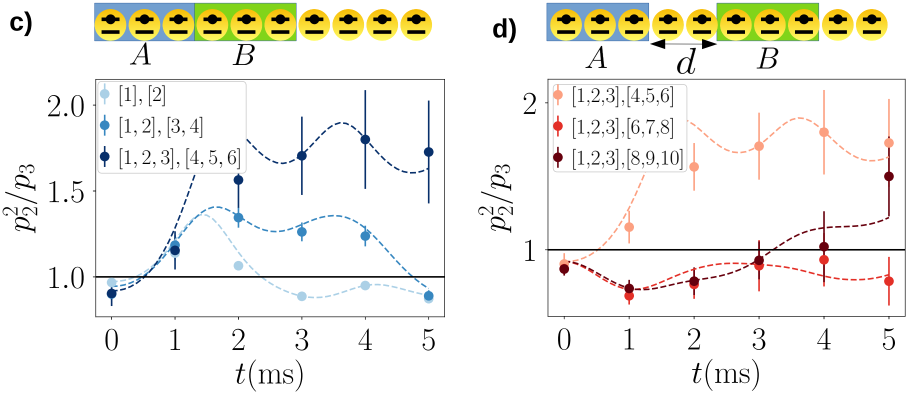


- Using the PPT condition: If $\exists\lambda_i < 0 \in \mathrm{Spec}(\rho^\Gamma)$ then A and B entangled.
- We propose the experimentally measurable p_3-PPT condition

$$\text{If } p_3 - p_2^2 = \sum_i \lambda_i(\lambda_i - p_2)^2 < 0 \text{ then } A \text{ and } B \text{ entangled.}$$

Experimental demonstration

- Data from Brydges et al, Science 2019, reanalyzed.



- Follow-ups conditions: Neven et al, NJP quantum info 2021, Yu et al, PRL 2021.
- Can we distinguish different kinds of entanglement structure?

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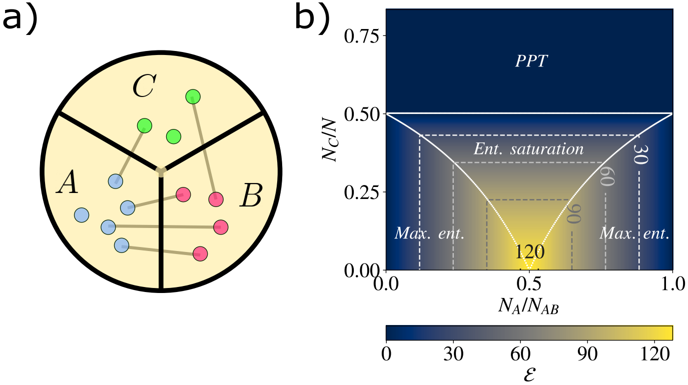
- Joint work with M. Votto, J. Carrasco, V. Vitale, C. Kokail, P. Zoller, and B. Kraus
- Can we use PT moments to experimentally probe an "entanglement phase diagram"?



Introducing r_2

- For a Haar random state made of $N_A + N_B + N_C$ qubits, we obtain the average PT moments of the reduced state $\rho = \mathrm{tr}_C(|\psi\rangle\langle\psi|)$ ($L_X = 2^{N_X}$)

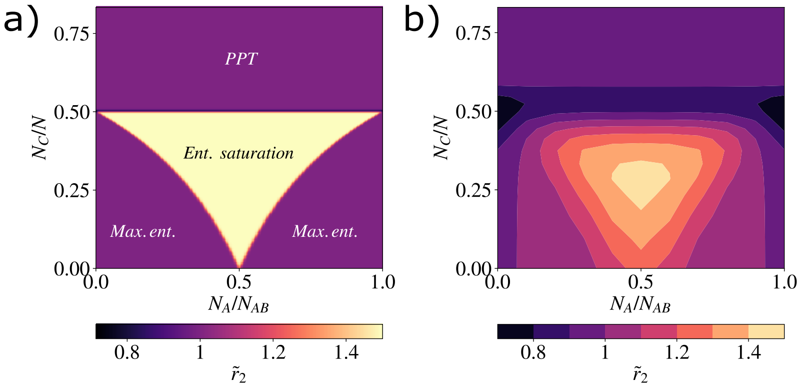
$$E[p_n] \simeq \frac{1}{(L_A L_B L_C)^n} \sum_{\tau \in S_n} L_C^{c(\tau)} L_A^{c(\sigma_+ \circ \tau)} L_B^{c(\sigma_- \circ \tau)},$$

- We define an ‘averaged’ ratio $\tilde{r}_2$

$$\tilde{r}_2 = \frac{E[p_2]E[p_3]}{E[p_4]}$$

r_2 reveals the entanglement phase diagram of random states

- r_2 is quantized in the thermodynamic limit $N \to \infty$



- Remark: How to do distinguish the Max-Ent Phase from the PPT phase?

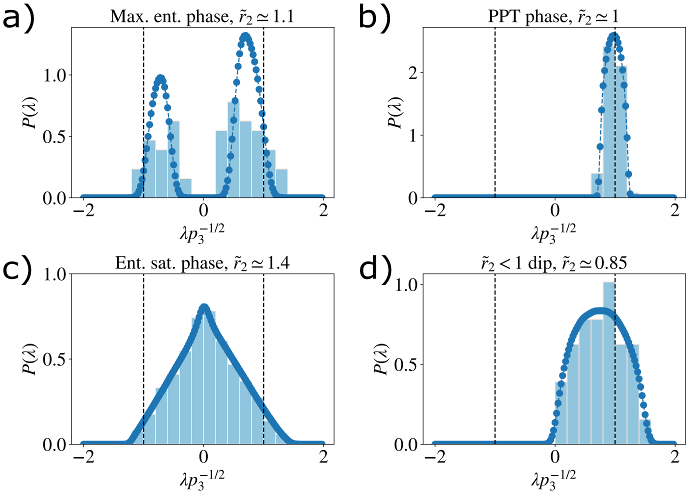
r_2 for individual quantum states

- Statistical fluctuations vanish in the thermodynamical limit. Thus, we can study $r_2 = (p_2 p_3)/p_4$ for individual quantum states.
- r_2 probes the 'shape' of the spectrum of ρ^Γ, $\{\lambda_i\}$.

$$r_2 = \frac{(\sum_i \lambda_i^2)(\sum_j \lambda_j^3)}{(\sum_k \lambda_k^4)}$$

- In particular, $\lambda_i^2 = p_3$ implies $r_2 = 1$.

r_2 for individual quantum states



A very sensitive test of randomness

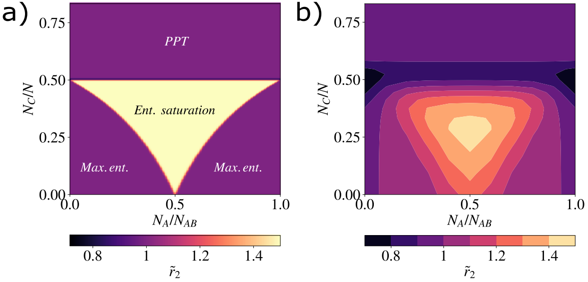
- Does r_2 make the difference between Haar random states and different states?
- First example: Clifford states generated by the n-qubit Clifford group (unitaries that map Pauli operators to Pauli operators).
- Clifford states can be highly entangled, but they are classically simulatable (Gottesman-Knill theorem)

A very sensitive test of randomness

- According to the 'Bravy' decomposition,

$$|\psi\rangle \quad = \quad U_A U_B U_C \, |0\rangle^{\otimes s_A} \, |0\rangle^{\otimes s_B} \, |0\rangle^{\otimes s_C} \, |\mathrm{GHZ}\rangle^{\otimes g_{ABC}} \, |\mathrm{EPR}\rangle^{\otimes e_{AB}} \, |\mathrm{EPR}\rangle^{\otimes e_{AC}} \, |\mathrm{EPR}\rangle^{\otimes e_{BC}} \, ,$$

  which implies $r_2 = 1$
- This implies the entanglement saturation phase $r_2 > 1$ is not Clifford simulatable



- We obtained similar conclusions for other classically simulatable states: fermionic Gaussian states, Matrix-Product-States.

Summary - Outlook

- PT moments provide practical detection criteria for mixed-state entanglement
- They can also distinguish different types of entanglement.
- What is the behavior of r_2 for non-integrable topological ordered states? is it different for an integrable model (toric code)?
- Can we simplify the experimental procedures to "test" Haar random states? eg use polynomial functionals instead of a ratio, that we can be measured with generalized swap tests.

Thank you!

The p_3 negativity complements r_2

- We define the p_3 negativity

$$\mathcal{E}_3 = \frac{1}{2} \log \left(\frac{p_2^2}{p_3} \right)$$

- p_3-PPT condition: $\mathcal{E}_3 > 0$ implies entanglement
- For Clifford states, $\mathcal{E}_3 = \mathcal{E}$.
- The p_3 ppt condition detects all entangled states in the thermodynamic limit!