Recent progress in the development of the randomized measurement toolbox

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Outline

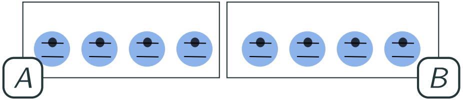
The randomized measurement toolbox for measuring entanglement

Extending the system sizes via importance sampling

Measuring new quantities

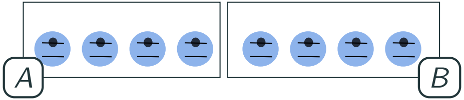
Entanglement

- Take two parts of a quantum system A B (eg sets of qubits)



- A and B are entangled iff $|\psi\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle$
- Example with two qubits. The Bell state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is entangled

How to quantify entanglement?

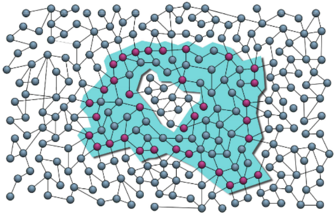


For pure states, entanglement entropies of the reduced state $\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$ are entanglement measures

$$\begin{aligned} S_{\mathrm{vN}} &= -\mathrm{Tr}[\rho_A \log(\rho_A)] \text{ von Neumann Entropies} \\ S_\alpha &= \frac{1}{1-\alpha} \log[\mathrm{Tr}(\rho_A^\alpha)] \text{ Rényi entropies} \end{aligned} \tag{1}$$

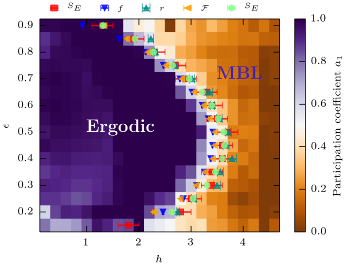
Why should I care about entanglement scaling?

For groundstates of condensed matter systems, we expect an area law $S_{\mathrm{vN}} \propto N_A^{(\mathrm{boundary})}$



Eisert RMP 2010

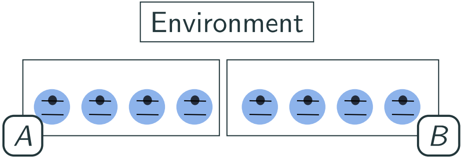
Entanglement scalings indicate phase transitions, a topological phase, etc



Luiz et al, PRB 2015

Entanglement in noisy quantum computers

- Typically the state of a quantum computer is 'mixed' due to an environment



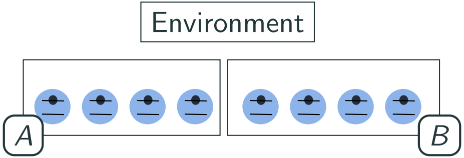
- We define the density matrix of the quantum computer

$$\rho_{AB} = \mathrm{Tr}_E(|\psi\rangle\langle\psi|) \tag{2}$$

- A and B are entangled iff

$$\rho_{AB} \neq \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)} \tag{3}$$

How to detect entanglement?



- Purity $\mathrm{Tr}(\rho^2)$ ($= 1$ iff the state is pure $\rho = |\psi\rangle\langle\psi|$)
- Purity entanglement condition (Horodecki 1996)

$$\mathrm{Tr}(\rho_A^2) < \mathrm{Tr}(\rho_{AB}^2) \implies A \text{ and } B \text{ are entangled} \tag{4}$$

- Example Bell State $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)|0_E\rangle$

$$\begin{aligned} \rho_{AB} &= \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|) \implies \mathrm{Tr}(\rho_{AB}^2) = 1 \\ \rho_A &= \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \implies \mathrm{Tr}(\rho_A^2) = 1/2 \end{aligned} \tag{5}$$

How to measure entanglement?

- The purity $\mathrm{Tr}(\rho^2)$ can be used to detect entanglement.
- The second Rényi entropy $S_2 = -\log_2[\mathrm{Tr}(\rho^2)]$ quantifies entanglement in quantum computers, in particular in the context of quantum simulation.

How to measure the purity in an experiment?

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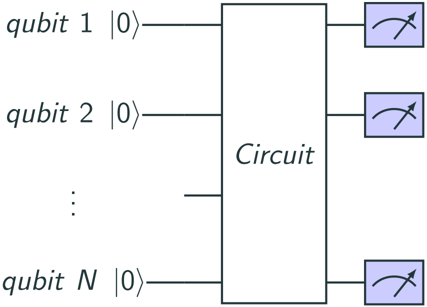
A standard measurement sequence in a quantum computer

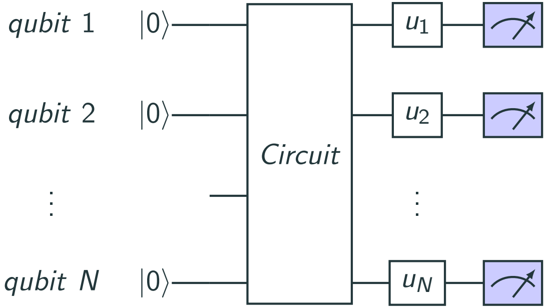


- Quantum measurements: the state $|s\rangle$ is measured with probability $\langle s|\rho|s\rangle$, in a certain measurement basis
- We have access to observables of the type $O\,|s\rangle = O(s)\,|s\rangle$.

$$\langle O\rangle = \sum_s \langle s|\rho|s\rangle\, O(s) \tag{6}$$

- Measurement in the 3^N combinations of X, Y, Z basis, I can realize state tomography, i.e measure ρ.
- Can I measure the purity $\mathrm{Tr}(\rho^2)$ more directly?

One approach: Randomized measurements



- Randomized measurements: We measure $P_u(s) = \langle s|u\rho u^\dagger|s\rangle$, $u = u_1 \otimes \cdots \otimes u_N$.
- We extract quantities of interest from the statistics of $P_u(s)$, over random unitary transformations.

Original protocol: van Enk-Beenakker (PRL 2012)

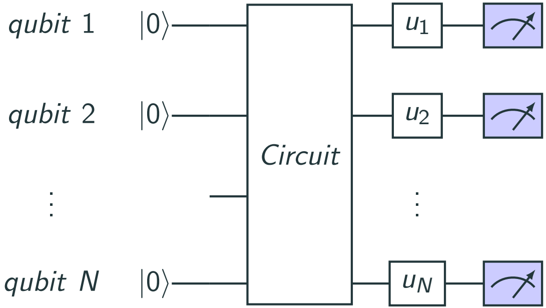
- Consider a single qubit

We measure the statistics of $P_u(s) = \langle s|u\rho u^\dagger|s\rangle$.

- Extreme case 1: The state is pure with $\mathrm{Tr}(\rho^2) = 1$, eg $\rho = |0\rangle\langle 0|$, then $P_u(s) = |\langle s|u|0\rangle|^2$ fluctuates in $[0, 1]$.
- Extreme case 2: The state is fully mixed with $\mathrm{Tr}(\rho^2) = 1/2$, $\rho = \mathbf{1}/2$, then $P_u(s) = 1/2$ does not fluctuate.
- Using the properties of the circular unitary ensemble (CUE)

$$(\mathrm{Tr}(\rho_A^2) - 1/2) \propto \mathrm{Var}_u[P_u(s)^2]) \tag{7}$$

RM protocol for qubits (A. Elben, BV et al, PRL 2018)



- Protocol:

  (i) apply independent random single qubit rotations

  (ii) measure states $|s\rangle = |s_1, \ldots, s_N\rangle$

  (iii) Postprocess data ($D(s, s')$ is the Hamming distance, number of mismatchs between s and s')

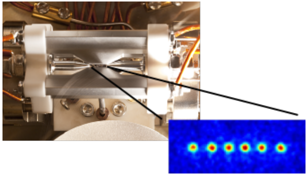
$$\mathrm{Tr}(\rho^2) = 2^N E_u\Big[\sum_{s,s'} (-2)^{-D(s,s')} P_u(s) P_u(s')\Big]$$

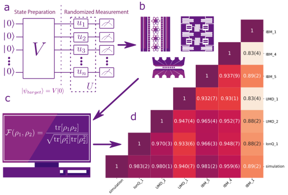
RM protocol for qubits (A. Elben, BV et al, PRL 2018)

$$\mathrm{Tr}(\rho^2) = 2^N E_u \Big[\sum_{s,s'} (-2)^{-D(s,s')} P_u(s) P_u(s') \Big]$$

- Proof: Random matrix theory and replica tricks.
- State-agnostic estimation without reconstructing the state
- Cheap postprocessing of the measurement data (ie no fitting, etc)
- Info on the unitaries does not appear in the formula $\rightarrow$ estimations are *robust*.
- Statistical errors $\rightarrow$ large required number of measurements $\sim 2^N$, but much smaller than for quantum state tomography $\sim 4^N$.

RMs are now used routinely in the lab





- Entropy measurements of up to 10 qubits (Brydges et al, Science 2019)
- Measurement of the topological entanglement entropy (Satzinger et al, Science 2021)
- Cross-Platform verification of devices (A. Elben, BV et al, PRL 2020) and (Zhu et al, 2021 preprint)

How can we improve the randomized measurement toolbox?

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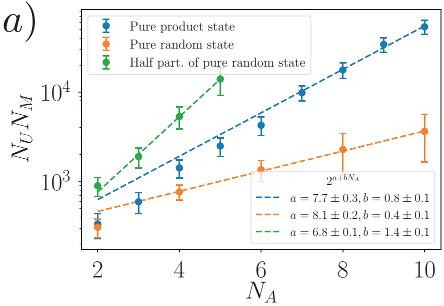
Statistical errors in RM protocols

- Main challenge for RM: statistical errors, with two crucial parameters who scale exponentially with system size
  - N_u number of applied unitaries
  - N_M number of measurement for each unitary



- Access to $\sim$ 15 qubits, assumption free, cost in postprocessing: few seconds.

Importance sampling for probing entanglement

- Work with A. Rath, A. Elben, R. van Bijnen, and P. Zoller (Phys. Rev. Lett. 2021)
- Our goal: reduce the required $N_U N_M$ exponentially (in particular N_u) $\rightarrow$ access to $30 - 35$ qubits.
- Why? Access universal regimes for entanglement: scaling laws, central charge, topological entropy, etc - Assess fundamental limits about measurements
- Our idea:
  - Importance Sampling: we use/learn information about the state *before* we measure
  - We *still* have unbiased estimators, i.e assumption-free, and cheap data postprocesing). We simply boost the convergence w.r.t statistical errors.

Importance sampling for probing entanglement: The basic idea

- Interpet RM as the evaluation of an integral

$$\mathrm{Tr}(\rho^2) = 2^N \int du \left[\sum_{s,s'} (-2)^{-D(s,s')} P_u(s) P_u(s')\right]$$

- Consider a 'well-chosen' probability distribution, instead of the uniform one

$$\mathrm{Tr}(\rho^2) = 2^N \int p_{\mathrm{IS}}(u) du \left[\frac{\sum_{s,s'} (-2)^{-D(s,s')} P_u(s) P_u(s')}{p_{\mathrm{IS}}(u)}\right]$$

- The unitaries u are sampled according to $p_{\mathrm{IS}}(u)$, this will change the convergence properties of the integral evaluation with finite number of samples, i.e measurements.
- Data acquisition and postprocessing task remain unchanged.

Importance sampling for probing entanglement: A concrete example

- Instead of the ideal pure N-qubit GHZ state $|\psi\rangle = (|0\rangle^{\otimes N} + |1\rangle^{\otimes N})/\sqrt{2}$, we realize a mixed-state version ρ of $|\psi\rangle$.
- To measure the purity of ρ, define the importance sampler

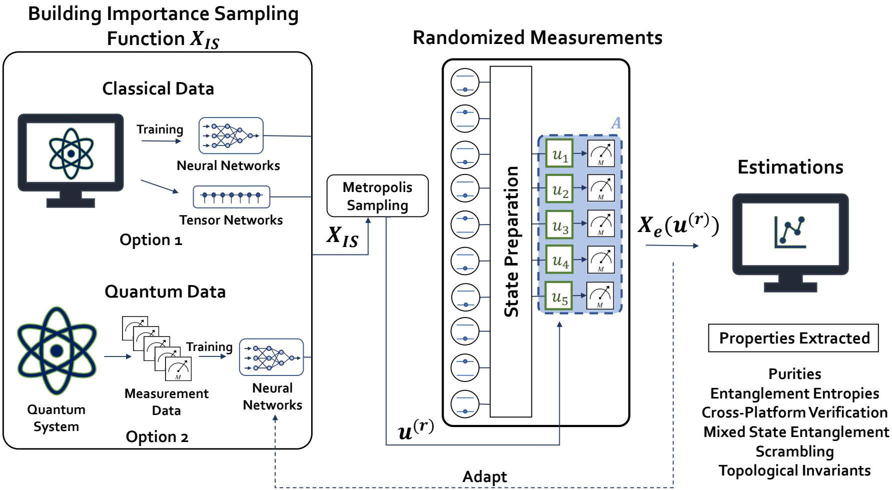
$$p_{\mathrm{IS}}(u) = 2^N \left[\sum_{s,s'} (-2)^{-D(s,s')} P_u^{(\psi)}(s) P_u^{(\psi)}(s') \right] \tag{8}$$

- Sample N_u unitaries according to $p_{\mathrm{IS}}(u)$ and estimate

$$[\mathrm{Tr}(\rho^2)]_e = \frac{1}{N_u} \sum_u \left[\frac{\sum_{s,s'} (-2)^{-D(s,s')} P_u(s) P_u(s')}{p_{\mathrm{IS}}(u)} \right] \tag{9}$$

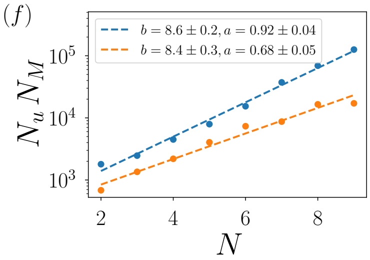
- As $\rho \approx |\psi\rangle\langle\psi|$, the integrand has been 'flattened' $\rightarrow$ small statistical fluctations

The full protocol



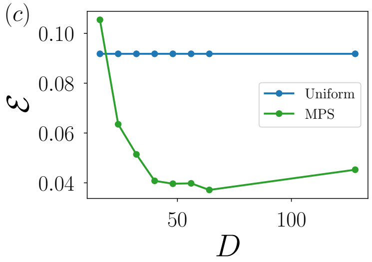
Performances

Example: GHZ states



→ Exponential reduction of the number of measurements, with $N_u = O(1)$.

Example: 10 *qubit* many-body entangled states



→ Better approximations lead to better performances

- Also tested on topological entropy of 2D topological ground states.
- Tutorial and python scripts: `https://github.com/bvermersch/RandomMeas`

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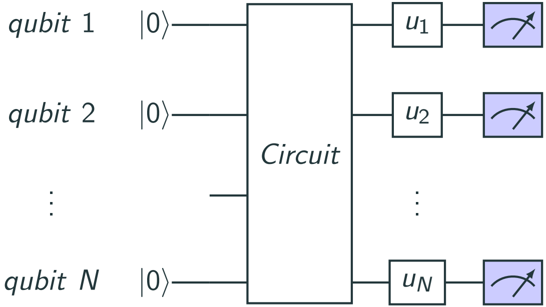
- Measurement of the Quantum Fisher information: Rath, Branciard, Minguzzi, Vermersch, Phys. Rev. Lett 2021
- Observation of the Entanglement barrier: Rath, Murciano, Vitale, Votto, Kueng, Dubail, Branciard, Calabrese, Vermersch, arXiv:2209.04393.

Measuring new quantities



- Important upgrade to the toolbox: Classical shadows (Huang et al, Nature Physics 2020)

$$\rho^{(r)} = \bigotimes_{i \in AB} \left(3 u_i \left| k_i^{(r)} \right\rangle \left\langle k_i^{(r)} \right| u_i^\dagger - \mathbf{1}_i \right) \tag{10}$$

- For a single measurement, unbiased estimations of the density matrix

$$E[\rho^{(r)}] = \rho \tag{11}$$

Measuring new quantities

- Classical shadows provide access to 'multi-copy observables' (MCO)

$$\mathrm{Tr}(O\rho^{\otimes n}) = E[\mathrm{Tr}(O\rho^{(r_1)} \otimes \cdots \otimes \rho^{(r_n)})] \tag{12}$$

- Examples: Rényi entropies, Partial-Transpose moments (Elben et al, PRL 2020), Symmetry-resolved entropies (Vitale et al, Sci Post 2021), etc
- Can I write/measure a given physical quantity as MCO? What is the cost in terms of statistical errors? postprocessing?

Quantum Fisher information

- Quantum Fisher information (QFI)

$$F_Q = 2 \sum_{(i,j),\lambda_i+\lambda_j>0} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j} |\langle i| A |j\rangle|^2 \text{ with } \rho = \sum_i \lambda_i |i\rangle\langle i| \tag{13}$$

- Certifies metrological power and entanglement depth (how many particles are entangled)
- Rath et al (PRL 2021): QFI is the limit of a series of MCO, i.e can be measured!

$$F_n = 2\,\mathrm{Tr}\left(\sum_{\ell=0}^{n} (\rho \otimes \mathbf{1} - \mathbf{1} \otimes \rho)^2 (\mathbf{1} \otimes \mathbf{1} - \rho \otimes \mathbf{1} - \mathbf{1} \otimes \rho)^\ell \mathbf{S}(A \otimes A) \right). \tag{14}$$

Quantum Fisher information and MCO: Statistical Errors

- The series of MCO F_n converges to QFI, and is estimated via randomized measurements

$$F_n = \mathrm{Tr}(O\rho^{\otimes n}) = E[\mathrm{Tr}(O\rho^{(r_1)} \otimes \cdots \otimes \rho^{(r_n)})] \tag{15}$$

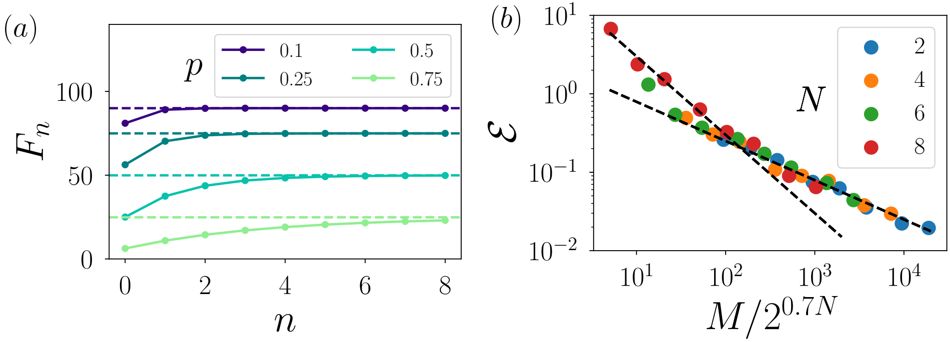
- What is the cost in measurements of a MCO O as a function of the order n?
- Typical number of measurement (error ϵ, confidence δ)

$$M \geq \max_{1\leq k\leq n} \left\{ \left(\frac{n\, n!^2}{k!(n-k)!^2} \frac{\mathrm{tr}([O_k]^2)}{\epsilon^2 \delta} \right)^{\frac{1}{k}} 2^N + k - 1 \right\}. \tag{16}$$

- $$O_k = \frac{1}{q!} \sum_{\pi} \mathrm{Tr}_{\{k+1\ldots n\}}(\pi^\dagger O \pi [\mathbf{1}^{\otimes k} \otimes \rho^{\otimes(n-k)}]) \tag{17}$$

Quantum Fisher information and MCO: Illustrations

- Illustration with a noisy GHZ state (depolarization noise p)



Quantum Fisher information and MCO: final remarks

- Complicated quantities such as QFI can be expressed as MCO and measured with a known measurement budget (!)
- The relation between the order n and the number of measurements M illustrates a tradeoff between the information given by a quantity w.r.t the cost to measure it!
- The measurement is not the only important aspect to take into account:
  - Postprocessing matters $\mathrm{E}\left[\mathrm{Tr}(O\rho^{(r_1)} \otimes \cdots \otimes \rho^{(r_n)})\right]$ supposes to evaluate $\mathcal{O}(M^n)$ terms
  - This can be avoided using ‘batch classical shadows’, arXiv:2209.04393, without altering statistical performances.

Conclusions

- Randomized measurements are an active field of research: experimentally friendly with plenty of non-trivial questions on the theory side
- Review: *The randomized measurement toolbox* A. Elben, S. T. Flammia, H.-Y. Huang, R. Kueng, J. Preskill, B. V, P. Zoller, arXiv:2203.11374