Quantum algorithms 2021/2022: Final exam

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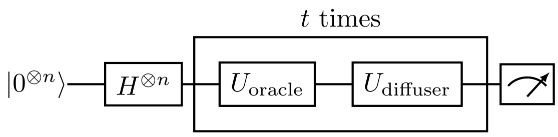
- The exam consists of two problems.
- Documents allowed: Slides of the lectures, documents of the exercices, hand-written notes
- You can only use your laptop to look at the documents from Moodle.
- You can also use printed versions of these documents.
- The use of smartphones or tablets is not allowed.

1 Grover's algorithm with multiple solutions ($\approx 7/20$)

We define a $n-$bit Boolean function $f(x)$, $x = (x_1, \ldots, x_n)$. We assume the existence of $M \geq 1$ distinct solutions $w_{m=1,\ldots,M}$, such that $f(w_m) = 1$.

Grover's algorithm is implemented via the following quantum circuit (cf Lecture 2):



with $U_{\text{oracle}} |x\rangle = (-1)^{f(x)} |x\rangle$, and $U_{\text{diffuser}} = 2 |\psi\rangle \langle\psi| - 1$, $|\psi\rangle = \frac{1}{\sqrt{N}} \sum_x |x\rangle$.

1. Write the state $|\psi_0\rangle$ before the first application of the oracle, as a function of $|\alpha\rangle = \frac{1}{\sqrt{N-M}} \sum_{x \neq w_m} |x\rangle$ and $|\beta\rangle = \frac{1}{\sqrt{M}} \sum_{w_m} |w_m\rangle$. You can introduce the angle θ, defined as $\sin(\theta/2) = \sqrt{M/N}$, $\cos(\theta/2) = \sqrt{(N-M)/N}$.
2. Write how the two states $|\alpha\rangle$, $|\beta\rangle$ are transformed after application of the diffuser.
3. Write the state $|\psi_{t=1}\rangle$ after the first iteration of the circuit.
4. Write the state $|\psi_t\rangle$ after an arbitrary number of iterations.
5. Express the probability p to measure a bitstring x that belongs to the set of solutions, i.e such that $x \in \{w_1, \ldots, w_M\}$, after t iterations. Why is it important here to know in advance the value of M? Simplify the expression in the limit of small $\theta \ll 1$ and large number of iterations $t \gg 1$.
6. Express the condition on the number of iterations t to observe a solution with high probability p. Express how such required value of t scales with N and M. Compare with the case $M = 1$ shown in Lecture 2.

2 Quantum error correction with the five qubit code ($\approx 13/20$)

The five qubit code is a quantum error correction code that uses five physical qubits to encode one logical qubit.

2.1 Defining the code

1. Explain the meaning of a physical, and of a logical qubit.
2. The five qubit code can be described in terms of a $[5, 1]$ stabilizer code, with the stabilizer group S generated by 4 elements

$$
\begin{aligned}
g_1 &= X_1 Z_2 Z_3 X_4 \\
g_2 &= X_2 Z_3 Z_4 X_5 \\
g_3 &= X_1 X_3 Z_4 Z_5 \\
g_4 &= Z_1 X_2 X_4 Z_5 .
\end{aligned}
$$

Show that such group S fulfills the conditions for being a stabilizer group.

3. Explain how one can *formally* define the code world $\{|0\rangle_L, |1\rangle_L\}$, i.e the Hilbert space of dimension 2 defining the logical qubit, based on the stabilizer group.

4. Calculate analytically the error syndromes for an error X_1 on the first qubit.

5. Without further calculations, list in a table the possible error syndromes for each qubit error. Show that each single qubit error can be detected and corrected.

6. Show that the following states can be used to define a logical qubit

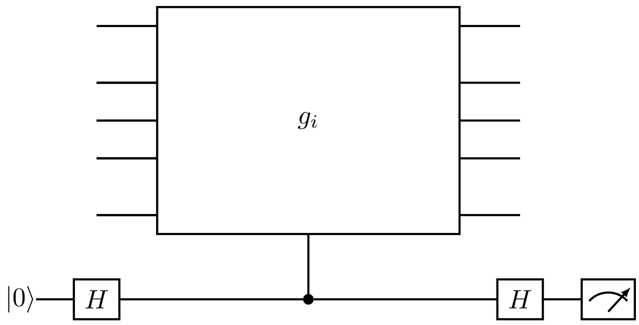
$$
\begin{aligned}
|0\rangle_L &= \prod_i (1+g_i)\,|0\rangle^{\otimes N} \\
|1\rangle_L &= \prod_i (1+g_i)\,|1\rangle^{\otimes N}
\end{aligned} \tag{1}
$$

7. Show that $Z_L = Z_1Z_2Z_3Z_4Z_5$ and $X_L = X_1X_2X_3X_4X_5$ can be used as single qubit logical gates. Write also the expression of the logical Y_L gate as a function of the physical qubits operators.

2.2 Stabilizer measurement

1. Explain why we need an ancilla qubit to perform an error syndrome.

2. We consider the following measurement circuit. We denote with $|\psi\rangle$ the wavefunction of the five physical qubits prior to the coupling to the ancilla qubit. Write the two probabilities $p(0)$, $p(1)$ to meaure the ancilla qubit in the state 0, and 1, respectively.



3. Show that we can write any g_i in terms of two projector operators $P_i(\pm 1)$, such that $g_i = P_i(1) - P_i(-1)$ with $P_i(1) + P_i(-1) = 1$.

4. Show that the ancilla measures the probabilities that the state $|\psi\rangle$ belongs to the eigenvalue ± 1 of the operator g_i, i.e that

$$
\begin{aligned}
p(0) &= |\langle\psi|\,P_i(1)\,|\psi\rangle|^2 \\
p(1) &= |\langle\psi|\,P_i(-1)\,|\psi\rangle|^2.
\end{aligned} \tag{2}
$$

Interpret this result: Does this circuit correctly perform an error syndrome?

5. Briefly explain how to write a full circuit for performing error detection and correction (I am not asking to write down the full circuit explicitely).

2.3 Encoding a quantum state

1. Explain a strategy to initialize a logical qubit $|0\rangle_L$ from the initial state $|0\rangle^{\otimes N}$ based on only performing error detection and correction.

2.4 Performance (Bonus questions)

We consider a probability p_e of error on each physical qubit, occuring independently.

1. For the 5 qubit code, express the probability that a logical qubit $|0\rangle_L$ undergoes an error which *cannot* be corrected.

2. Write the condition on p_e to achieve 'useful' quantum error correction, i.e to obtain that the 5 qubit code performs better than a single physical qubit. Do we satisfy this condition in the limit of small values of $p_e \to 0$?