Quantum algorithms 2022/2023: Final exam

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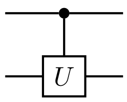
- Documents allowed: Slides of the lectures, documents of the exercices, hand-written notes
- You can only use your laptop to look at the documents from Moodle.
- You can also use printed versions of these documents.
- The use of smartphones or tablets is not allowed.

1 Warm-up on controlled-U operations

1. We consider a unitary matrix U that acts on m qubits. The controlled-U operation



acts on $1+m$ qubits, and is defined by the transformation

$$C[U] = |0\rangle\langle 0| \otimes \mathbf{1}_m + |1\rangle\langle 1| \otimes U. \tag{1}$$

Give an example of a controlled-U operation that was studied during the lecture for $m=1$, and write the corresponding output states $|\psi'\rangle$ for all the possible initial states $|\psi\rangle = |00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$.

2. Let $|\psi\rangle = |x_1, y_1, \dots, y_m\rangle$ an input state for the controlled-U operation. Show that

$$C[U]\,|\psi\rangle = |x_1\rangle \otimes U^{x_1}\,|y_1, \dots, y_m\rangle\,. \tag{2}$$

3. Write the state of the system $C[U]\,|\psi\rangle$ for input state of the form

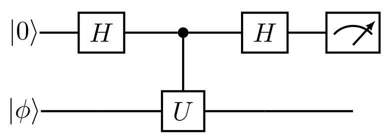
$$|\psi\rangle = \sum_{x_1, y_1, \dots, y_m} c_{x_1, y_1, \dots, y_m}\,|x_1, y_1, \dots, y_m\rangle \tag{3}$$

2 Quantum Phase Estimation with one qubit measurements

1. We consider a unitary matrix U acting on m qubits, and an eigenstate $|\phi\rangle$ of U. Show that we can define a real number $\delta \in [0, 1[$ such that

$$U\,|\phi\rangle = e^{2i\pi\delta}\,|\phi\rangle \tag{4}$$

2. The goal of the quantum phase estimation algorithm (QPE) is to estimate the phase δ. We first consider a simplified version of the QPE using $1+m$ qubits. The first qubit is initialized in $|0\rangle$, the last m qubits are prepared in the eigenstate $|\phi\rangle$. Then, we apply the following circuit.



Write the wavefunction of the system after the first Hadamard gate

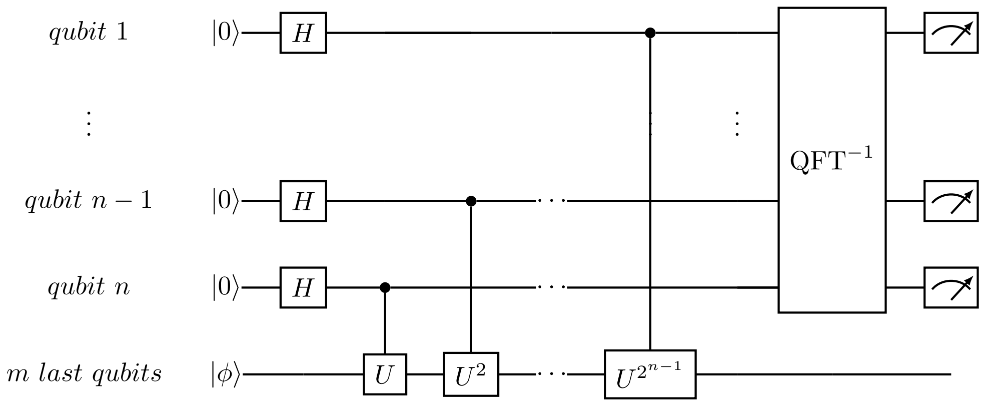
3. Write the wavefunction of the system as a function of δ after the controlled-U operation.

4. Write the wavefunction of the system after the last Hadamard gate.

5. We assume here that the phase can be either $\delta = 0$ or $\delta = 1/2$. Show that a single measurement of the first qubit allows us to extract δ with unit probability.

3 Quantum phase estimation with n-qubit measurements

We consider the general QPE algorithm that uses a circuit of $n+m$ qubits. The choice of $n \geq 1$ controls the 'resolution' in determining δ, while the unitary U still acts on m qubits.



The first n qubits are first subject to a Hadamard gate, then each qubit $j = 1, \ldots, n$ controls a $U^{2^{n-j}}$ operation. At the end, the inverse quantum Fourier transform is applied before measurement. Again, note that the last m qubits are initialized in the eigenstate $|\phi\rangle$ of U.

1. Write the state of the system after the Hadamard gates. As in the lectures, we will use the notation $|x\rangle = |x_1, \ldots, x_n\rangle$, $x = \sum_{j=1}^{n} x_j 2^{n-j}$ to denote the 2^n basis states of the n-qubit system.

2. Consider now one of the controlled operations $C_j[U^{2^{n-j}}]$ that uses the qubit j as controlled qubit

$$C_j[U^{2^{n-j}}] = |0\rangle_j \langle 0| \otimes \mathbf{1}_m + |1\rangle_j \langle 1| \otimes U^{2^{n-j}}. \tag{5}$$

   Show that the action on the state $|x\rangle |\phi\rangle$ can be written as

$$C_j[U^{2^{n-j}}] |x\rangle |\phi\rangle = |x\rangle \otimes (U^{(2^{n-j})x_j} |\phi\rangle) \tag{6}$$

3. Write the state of the system after the last controlled operation $C_1[U^{2^{n-1}}]$, as a function of δ and x

4. We recall the expression of the inverse quantum Fourier transform operation

$$QFT^{-1} |x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} e^{-2i\pi xy/2^n} |y\rangle \tag{7}$$

   Write the expression of the final state before measurement. You can define the amplitude c_y such that $|\psi\rangle = \sum_{y=0}^{2^n-1} c_y |y\rangle |\phi\rangle$.

5. We assume that the phase δ can be written as $\delta = s/2^n$, with $0 \leq s \leq 2^{n-1}$ an integer. What is the probability to reveal the correct phase from a single measurement?

6. We consider now the situation when $\delta 2^n$ is not an integer. Show that the probability to observe y can be written as

$$P(y) = \frac{1}{4^n} \frac{\sin^2(\pi 2^n(\delta - \tilde{y}))}{\sin^2(\pi(\delta - \tilde{y}))} \tag{8}$$

   with $\tilde{y} = y/2^n$.

7. Describe qualitatively the shape of $P(y)$, and explain why n controls the accuracy of the estimation of the phase δ.