

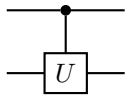
# Quantum algorithms 2022/2023: Final exam

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- Documents allowed: Slides of the lectures, documents of the exercices, hand-written notes
- You can only use your laptop to look at the documents from Moodle.
- You can also use printed versions of these documents.
- The use of smartphones or tablets is not allowed.

## 1 Warm-up on controlled- $U$ operations

1. We consider a unitary matrix  $U$  that acts on  $m$  qubits. The controlled- $U$  operation



acts on  $1 + m$  qubits, and is defined by the transformation

$$C[U] = |0\rangle\langle 0| \otimes \mathbf{1}_m + |1\rangle\langle 1| \otimes U. \quad (1)$$

Give an example of a controlled- $U$  operation that was studied during the lecture for  $m = 1$ , and write the corresponding output states  $|\psi'\rangle$  for all the possible initial states  $|\psi\rangle = |00\rangle, |01\rangle, |10\rangle, |11\rangle$ .

**Solution:** The  $CNOT$  is a controlled  $X$  operation acting  $1 + m = 2$  qubits. The respective output states are  $|00\rangle, |01\rangle, |11\rangle, |10\rangle$ .

2. Let  $|\psi\rangle = |x_1, y_1, \dots, y_m\rangle$  an input state for the controlled- $U$  operation. Show that

$$C[U]|\psi\rangle = |x_1\rangle \otimes U^{x_1} |y_1, \dots, y_m\rangle. \quad (2)$$

**Solution:** If  $x_1 = 0$ , then

$$C[U]|\psi\rangle = |x_1, y_1, \dots, y_m\rangle = |x_1\rangle \otimes U^0 |y_1, \dots, y_m\rangle. \quad (3)$$

else for  $x_1 = 1$

$$C[U]|\psi\rangle = |x_1\rangle \otimes |y_1, \dots, y_m\rangle = |x_1\rangle \otimes U^1 |y_1, \dots, y_m\rangle. \quad (4)$$

3. Write the state of the system  $C[U]|\psi\rangle$  for input state of the form

$$|\psi\rangle = \sum_{x_1, y_1, \dots, y_m} c_{x_1, y_1, \dots, y_m} |x_1, y_1, \dots, y_m\rangle \quad (5)$$

**Solution:** By linearity

$$C[U]|\psi\rangle = \sum_{x_1, y_1, \dots, y_m} c_{x_1, y_1, \dots, y_m} C[U]|x_1, y_1, \dots, y_m\rangle = \sum_{x_1, y_1, \dots, y_m} c_{x_1, y_1, \dots, y_m} |x_1\rangle \otimes U^{x_1} |y_1, \dots, y_m\rangle. \quad (6)$$

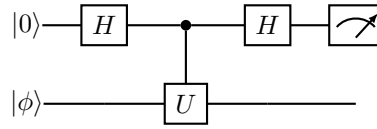
## 2 Quantum Phase Estimation with one qubit measurements

1. We consider a unitary matrix  $U$  acting on  $m$  qubits, and an eigenstate  $|\phi\rangle$  of  $U$ . Show that we can define a real number  $\delta \in [0, 1[$  such that

$$U|\phi\rangle = e^{2i\pi\delta} |\phi\rangle \quad (7)$$

**Solution:**  $|\phi\rangle$  is an eigenstate  $U|\phi\rangle = \epsilon|\phi\rangle$ .  $U^\dagger U = 1$  because  $U$  is unitary. Therefore,  $\langle\phi|U^\dagger U|\phi\rangle = |\epsilon|^2 = 1$ . This implies that, without loss of generality, I can define  $\delta$  as above.

2. The goal of the quantum phase estimation algorithm (QPE) is to estimate the phase  $\delta$ . We first consider a simplified version of the QPE using  $1 + m$  qubits. The first qubit is initialized in  $|0\rangle$ , the last  $m$  qubits are prepared in the eigenstate  $|\phi\rangle$ . Then, we apply the following circuit.



Write the wavefunction of the system after the first Hadamard gate

**Solution:**

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) |\phi\rangle \quad (8)$$

- Write the wavefunction of the system as a function of  $\delta$  after the controlled- $U$  operation.
- Write the wavefunction of the system after the last Hadamard gate.

**Solution:** After the  $C_U$

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle e^{2i\pi\delta}) |\phi\rangle \quad (9)$$

After the last Hadamard

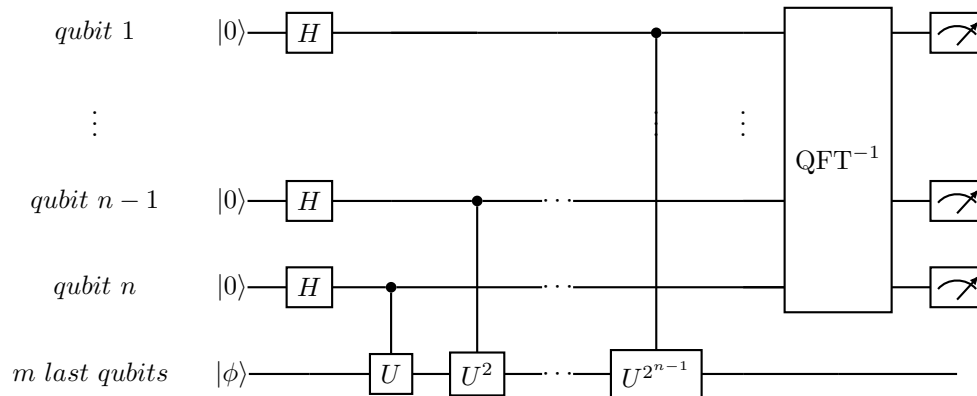
$$|\psi\rangle = \frac{1}{2} (|0\rangle + |1\rangle + e^{2i\pi\delta} [|0\rangle - |1\rangle]) |\phi\rangle \quad (10)$$

- We assume here that the phase can be either  $\delta = 0$  or  $\delta = 1/2$ . Show that a single measurement of the first qubit allows us to extract  $\delta$  with unit probability.

**Solution:** We obtain the measurement probability  $p(0) = 1$  if  $\delta = 0$ , and  $p(1) = 1$  if  $\delta = 1/2$ . This means a single measurement is sufficient to reveal the phase.

### 3 Quantum phase estimation with $n$ -qubit measurements

We consider the general QPE algorithm that uses a circuit of  $n + m$  qubits. The choice of  $n \geq 1$  controls the ‘resolution’ in determining  $\delta$ , while the unitary  $U$  still acts on  $m$  qubits.



The first  $n$  qubits are first subject to a Hadamard gate, then each qubit  $j = 1, \dots, n$  controls a  $U^{2^{n-j}}$  operation. At the end, the inverse quantum Fourier transform is applied before measurement. Again, note that the last  $m$  qubits are initialized in the eigenstate  $|\phi\rangle$  of  $U$ .

- Write the state of the system after the Hadamard gates. As in the lectures, we will use the notation  $|x\rangle = |x_1, \dots, x_n\rangle$ ,  $x = \sum_{j=1}^n x_j 2^{n-j}$  to denote the  $2^n$  basis states of the  $n$ -qubit system.

**Solution:**

$$|\psi\rangle = \frac{1}{\sqrt{2^n}} (|0\rangle + |1\rangle)^{\otimes n} |\phi\rangle = \frac{1}{\sqrt{2^n}} \sum_{x_1, \dots, x_n=0,1} |x_1, \dots, x_n\rangle |\phi\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle |\phi\rangle \quad (11)$$

- Consider now one of the controlled operations  $C_j[U^{2^{n-j}}]$  that uses the qubit  $j$  as controlled qubit

$$C_j[U^{2^{n-j}}] = |0\rangle_j \langle 0| \otimes \mathbf{1}_m + |1\rangle_j \langle 1| \otimes U^{2^{n-j}}. \quad (12)$$

Show that the action on the state  $|x\rangle |\phi\rangle$  can be written as

$$C_j[U^{2^{n-j}}] |x\rangle |\phi\rangle = |x\rangle \otimes (U^{(2^{n-j})x_j} |\phi\rangle) \quad (13)$$

**Solution:** By definition

$$C_j[U^{2^{n-j}}] = |0\rangle_j \langle 0| \mathbf{1} + |1\rangle_j \langle 1| U^{2^{n-j}} \quad (14)$$

We obtain if  $x_j = 0$

$$C_j[U^{2^{n-j}}] |x\rangle |\phi\rangle = |x\rangle |\phi\rangle \quad (15)$$

and if  $x_j = 1$

$$C_j[U^{2^{n-j}}] |x\rangle |\phi\rangle = |x\rangle \otimes (U^{2^{n-j}} |\phi\rangle) \quad (16)$$

3. Write the state of the system after the last controlled operation  $C_1[U^{2^{n-1}}]$ , as a function of  $\delta$  and  $x$

**Solution:** After the first controlled operation

$$|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_x |x\rangle U^{x_n} |\phi\rangle \quad (17)$$

After the last one

$$\begin{aligned} |\psi\rangle &= \frac{1}{\sqrt{2^n}} \sum_x |x\rangle \otimes (U^{x_1 2^{n-1} + \dots + x_n}) |\phi\rangle = \frac{1}{\sqrt{2^n}} \sum_x e^{2i\pi\delta(x_1 2^{n-1} + \dots + x_n)} |x\rangle |\phi\rangle \\ &= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} e^{2i\pi x\delta} |x\rangle |\phi\rangle \end{aligned} \quad (18)$$

4. We recall the expression of the inverse quantum Fourier transform operation

$$QFT^{-1} |x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} e^{-2i\pi xy/2^n} |y\rangle \quad (19)$$

Write the expression of the final state before measurement. You can define the amplitude  $c_y$  such that  $|\psi\rangle = \sum_{y=0}^{2^n-1} c_y |y\rangle |\phi\rangle$ .

**Solution:** We obtain the final state before measurement

$$|\psi\rangle = \frac{1}{2^n} \sum_{x,y=0}^{2^n-1} e^{2i\pi x(\delta - y/2^n)} |y\rangle |\phi\rangle = \sum_{y=0}^{2^n-1} c_y |y\rangle |\phi\rangle \quad (20)$$

with  $c_y = 1/(2^n) \sum_{x=0}^{2^n-1} \alpha_y^x$ ,  $\alpha_y = e^{2i\pi(\delta - y/2^n)}$ .

5. We assume that the phase  $\delta$  can be written as  $\delta = s/2^n$ , with  $0 \leq s \leq 2^n-1$  an integer. What is the probability to reveal the correct phase from a single measurement?

**Solution:** Consider the state  $y = s$ , we have  $\alpha_s = 1$ ,  $c_s = 1$  (which implies  $c_{y \neq s} = 0$  as the state is normalized). Thus the probability to measure the right outcome  $y = s$  is  $P(y = s) = 1$  ( $= 0$  for  $y \neq s$ ). This means a single measurement gives us the correct result  $s$ .

6. We consider now the situation when  $\delta 2^n$  is not an integer. Show that the probability to observe  $y$  can be written as

$$P(y) = \frac{1}{4^n} \frac{\sin^2(\pi 2^n (\delta - \tilde{y}))}{\sin^2(\pi (\delta - \tilde{y}))} \quad (21)$$

with  $\tilde{y} = y/2^n$ .

**Solution:** Using the expression of  $\alpha_y$ , we have

$$\begin{aligned} c_y &= \frac{1}{2^n} \frac{1 - \alpha_y^{2^n}}{1 - \alpha_y} \\ &= \frac{1}{2^n} \frac{\sin(\pi 2^n (\delta - \tilde{y}))}{\sin(\pi (\delta - \tilde{y}))} \end{aligned} \quad (22)$$

and therefore

$$P(y) = |c_y|^2 = \dots \quad (23)$$

7. Describe qualitatively the shape of  $P(y)$ , and explain why  $n$  controls the accuracy of the estimation of the phase  $\delta$ .

**Solution:** The function  $P$  is maximal when  $\tilde{y} \approx \delta$ . As  $n$  increases, the width of the peak narrows, i.e we can approximate  $\delta$  better and better (The statement can be made more quantitative, for instance by Taylor expanding  $P(y)$  around  $\delta$ ).