Quantum Error Correction (QEC)
A qubit in the real world

Consider a single qubit state

\[ |\psi\rangle = \alpha |0\rangle + \beta |1\rangle \]

Due to decoherence (e.g. spontaneous emission)

\[ |\psi\rangle \otimes |E\rangle \rightarrow \sqrt{1 - p_x - p_y - p_z} |\psi\rangle \otimes |E\rangle + \sqrt{p_x} X |\psi\rangle \otimes |E_x\rangle + \sqrt{p_y} Y |\psi\rangle \otimes |E_y\rangle + \sqrt{p_z} Z |\psi\rangle \otimes |E_z\rangle \]

No errors  Bit flip  Bit+Phase flip  Phase flip

How can we reset the qubit states (without destroying the qubit superposition)?
Quantum Error Correction

The only known way to do quantum error correction is to encode a **logical** qubit in an enlarged Hilbert space, e.g., many **physical** qubits.

Our first QEC code: **The three-qubit bit-flip code**

\[
|\psi\rangle = \alpha |000\rangle + \beta |111\rangle
\]

‘Logical 0’ \[|0\rangle_L\]

‘Logical 1’ \[|1\rangle_L\]
Three-qubit bit flip code

Our first QEC code: **The three-qubit bit-flip code** \( |\psi\rangle = \alpha |000\rangle + \beta |111\rangle \)

The basic idea:

One spin flips, e.g \( |\psi\rangle = \alpha |100\rangle + \beta |011\rangle \)

**First step → Error syndrome**: \( S = Z_1Z_2, Z_2Z_3 \)

→ measurement that does not destroy the superposition (\( |\psi\rangle \) eigenstate of the corresponding observable)

→ The measurements detects the error unambiguously

\[
\langle Z_1 Z_2 \rangle = -1 \quad \langle Z_2 Z_3 \rangle = 1
\]

**Second step → Error recovery**: \( X_1 |\psi\rangle \)

**Question**: Error syndrome/Recovery for the second and third qubit?
Our first QEC code: **The three-qubit bit-flip code**

**Implementation:**

1. Create a logical qubit via two qubit entangling gates

\[
|\psi\rangle = (\alpha |0\rangle + \beta |1\rangle) |00\rangle \\
|\psi\rangle = \alpha |000\rangle + \beta |111\rangle
\]

![Diagram](image-url)
Our first QEC code: **The three-qubit bit-flip code**

**Implementation:**

2. Evolve the logical qubit in the ‘code world’

**Question:** Write down the state at this point
Our first QEC code: **The three-qubit bit-flip code**

**Implementation:**

3. Error syndrome (without destroying the qubit superposition, i.e., measuring $Z_1, Z_2, Z_3$)

**Question:**
Why does this measure $\langle Z_1 Z_2 \rangle$ and $\langle Z_2 Z_3 \rangle$?

Can we correct for phase-flip errors?

4. Error recovery (easy part and optional)
Our second QEC code: The steane code

**Goal**: Correct arbitrary single qubit gates

\[ |\psi\rangle \otimes |E\rangle \rightarrow \sqrt{1 - p_x - p_y - p_z} |\psi\rangle \otimes |E\rangle + \sqrt{p_x} X |\psi\rangle \otimes |E_x\rangle + \sqrt{p_y} Y |\psi\rangle \otimes |E_y\rangle + \sqrt{p_z} Z |\psi\rangle \otimes |E_z\rangle \]

- No errors
- Bit flip
- Bit+Phase flip
- Phase flip

For the qubit, this error decomposition is complete (proof in terms of Kraus representation of quantum processes)

The Steane code corrects for bit flip, phase flips, and bit+phase flips, thus for arbitrary single qubit errors.
The Steane code

Code world

\[ |\psi\rangle = \alpha |0\rangle_L + \beta |1\rangle_L \]

\[ |0\rangle_L = |0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle \]
\[ + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle \]

\[ |1\rangle_L = X_{111111} |0\rangle_L . \]

→ 1 logical qubits = 7 physical qubits
The Steane code

\[ |0\rangle_L = |0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle \]

\[ |1\rangle_L = X_{111111} |0\rangle_L . \]

| Syndrome \( \text{Error} \) | 0 | X₁ | X₂ | X₃ | X₄ | X₅ | X₆ | X₇ | Z₁ | Z₂ | Z₃ | Z₄ | Z₅ | Z₆ | Z₇ | Y₁ | ...
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**Strict Conditions for QEC:**

→ (1) The code world is not affected by the syndrome measurement
→ (2) A unique error syndrome per error
Stabilizer formalism

Strict Conditions for QEC:
→ (1) Any logical qubit state is not affected by the syndrome measurement
→ (2) A unique error syndrome per error

These conditions can be easily checked for codes that are written in the stabilizer formalism

Stabilizer code $[n,k]$ (Gottesman and coworkers, 1997)

→ Consider $n$ physical qubits.
→ Consider $S$ a subgroup in the group of Pauli matrices generated by $(n-k)$ commuting elements $g_1, \ldots, g_{n-k}$
→ Then $V_S$, the vector space stabilized by $S$, is of dimension $2^k$, i.e. can encode $k$ logical qubits
→ The set of possible Errors $\{E_k\}$ that can be corrected are such that for any $j,k$
   (1) $E_j^{\text{dag}}E_k$ is in $S$
   Or (2) $E_j^{\text{dag}}E_k$ anticommutes with one element of $S$
**Stabilizer formalism**

**Stabilizer code \([n,k]\)**

- \(V_S\), the vector space stabilized by \(S\), is of dimension \(2^k\), i.e. can encode \(k\) logical qubits
- The set of possible Errors \(E=\{E_k\}\) that can be corrected are such that for any \(j,k\)
  1. \(E_j^\dagger E_k\) is in \(S\)
  2. \(E_j^\dagger E_k\) anticommutes with one element of \(S\)

**QEC recipe:** The code world is given by two orthonormal state of the vector space \(V_S\)
- Error Syndromes: Stabilizer measurements

**Example**  **The Bit-flip code is a \([3,1]\) stabilizer code!**

- \(E=\{I, X_1, X_2, X_3\}\) can be corrected by measuring the stabilizer generators \(S=\{Z_1Z_2, Z_2Z_3\}\)
- The logical states are \(|000\rangle\), \(|111\rangle\) belong to the vector space \(V_S\)

**Exercice**  Prove that the Steane code is a \([7,1]\) stabilizer code.
A single QEC code does not protect against any error. (ex error $X_1 X_2$ in the bit flip code.)

However, we may concatenate/combine several QEC codes to fight for error propagation provided the error probability per gate is below a certain threshold [Michael Ben-Or and Dorit Aharonov]

Fault-tolerant quantum computing = A significant technological challenge.
Estimation : ~10 000 physical qubits per logical qubit for the surface code

J. Preskill « I've already emphasized repeatedly that it will probably be a long time before we have fault-tolerant quantum computers solving hard problems. »
Towards fault-tolerance with the surface code

Physical qubits

Ancilla qubits → perform stabilizer measurements
All $Z_a Z_b Z_c Z_d$ and $X_a X_b X_c X_d$ (commuting operators)

Towards fault-tolerance with the surface code

Error syndrome = pattern of stabilization results

- **phase flip**
- **bit flip**
Towards fault-tolerance with the surface code

Initialization and operation

1) Measure all stabilizers → Creates a logical state

2) Apply logical operations $X_L, Z_L$ (Preserve the code world)

3) Check for errors by stabilizer measurements
Towards fault-tolerance with the surface code

Robustness of the logical qubit

QEC fails when the numbers of errors~system size $d/2$

Example: One error syndrome for two possible errors

Fault-tolerance can be achieved by increasing the array size
Towards fault-tolerance with the surface code

Experimental implementation of a minimal surface code

Stabilizer: $Z_{D_1}Z_{D_3}, Z_{D_2}Z_{D_4}$, and $X_{D_1}X_{D_2}X_{D_3}X_{D_4}$

$|0\rangle_L = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$

$|1\rangle_L = \frac{1}{\sqrt{2}}(|0101\rangle + |1010\rangle)$

Towards fault-tolerance with the surface code

Preparation of the $0_L$ logical state

→ start from $0^4$
→ measure all ancillas in 0

Repeted error correction
(postprocessing on the no-error signal)
Consequence: we need more physical qubits

### Scaling IBM Quantum technology

<table>
<thead>
<tr>
<th>IBM Q System One (released)</th>
<th>(In development)</th>
<th>Next family of IBM Quantum systems</th>
<th>and beyond</th>
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<tbody>
<tr>
<td>2019</td>
<td>2020</td>
<td>2021</td>
<td>2023</td>
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<td>27 qubits</td>
<td>65 qubits</td>
<td>127 qubits</td>
<td>1,121 qubits</td>
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<tr>
<td><em>Falcon</em></td>
<td><em>Hummingbird</em></td>
<td><em>Eagle</em></td>
<td><em>Condor</em></td>
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<td><strong>Path to 1 million qubits and beyond</strong></td>
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*Large scale systems*
Consequence: we need more physical qubits
QEC is a hot topic in modern quantum computing and a key challenge for quantum technologies.

Fault-tolerant quantum computing requires many physical qubit per logical qubit.

Surface codes implement QEC as 2D spin lattice models, which we also encounter in quantum simulation: Lecture 4.