Lecture 3

Quantum Error Correction (QEC)

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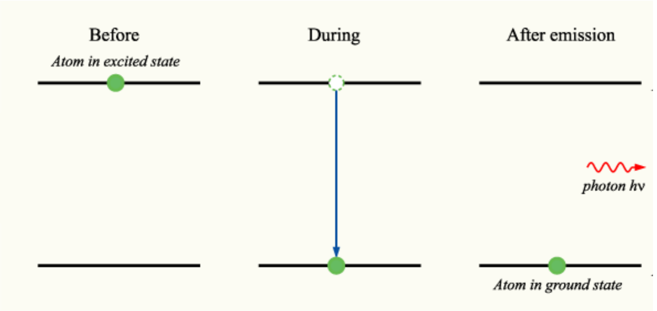
Lecture 3

A qubit in the real world

Consider a single qubit state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Due to decoherence (e.g. spontaneous emission)



$$|\psi\rangle\otimes|E\rangle \rightarrow \sqrt{1-p_x-p_y-p_z}\,|\psi\rangle\otimes|E\rangle+\sqrt{p_x}X\,|\psi\rangle\otimes|E_x\rangle+\sqrt{p_y}Y\,|\psi\rangle\otimes|E_y\rangle+\sqrt{p_Z}Z\,|\psi\rangle\otimes|E_z\rangle$$

Environnement

No errors

Bit flip

Bit+Phase flip

Phase flip

How can we reset the qubit states (without destroying the qubit superposition) ?

Quantum Error Correction

The only known way to do quantum error correction is to encode a **logical** qubit in a enlarged Hilbert space, e.g., many **physical** qubits

Our first QEC code: **The three-qubit bit-flip code**

$$|\psi\rangle = \alpha\,|000\rangle + \beta\,|111\rangle$$

'Logical 0' $|0\rangle_L$

'Logical 1' $|1\rangle_L$

Three-qubit bit flip code

Our first QEC code: **The three-qubit bit-flip code** $|\psi\rangle = \alpha\,|000\rangle + \beta\,|111\rangle$

The basic idea:

One spin flips, e.g $|\psi\rangle = \alpha\,|100\rangle + \beta\,|011\rangle$

First step → Error syndrome : S=Z_1Z_2, Z_2Z_3

→ measurement that does not destroy the superposition ($|\psi\rangle$ eigenstate of the corresponding observable)

→ The measurements detects the error unambiguously

$$\langle Z_1 Z_2\rangle = -1 \qquad \langle Z_2 Z_3\rangle = 1$$

Second step → Error recovery : $X_1\,|\psi\rangle$

Question: Error syndrome/Recovery for the second and third qubit ?

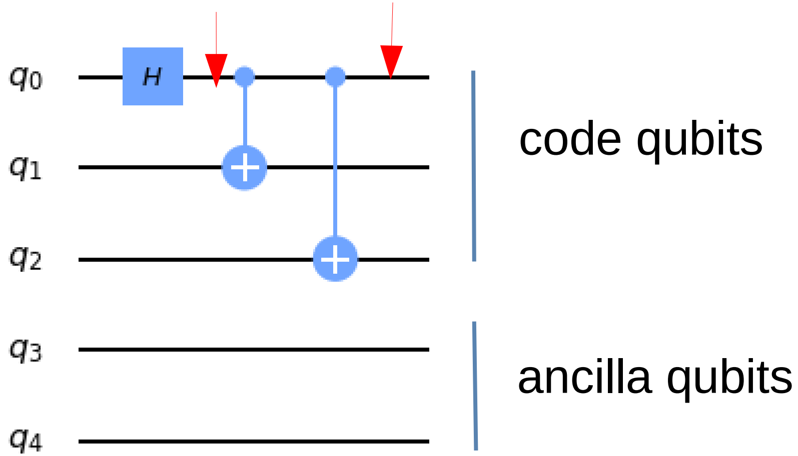
Three-qubit bit flip code

Our first QEC code: **The three-qubit bit-flip code**

Implementation :

1. Create a logical qubit via two qubit entangling gates

$$|\psi\rangle = (\alpha|0\rangle + \beta|1\rangle)|00\rangle \quad |\psi\rangle = \alpha|000\rangle + \beta|111\rangle$$

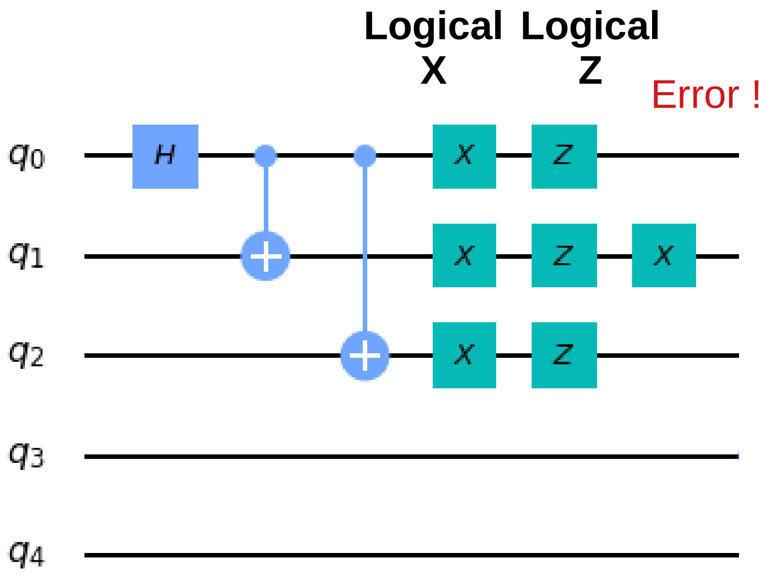


Three-qubit bit flip code

Our first QEC code: **The three-qubit bit-flip code**

Implementation :

2. Evolve the logical qubit in the ‘code world’



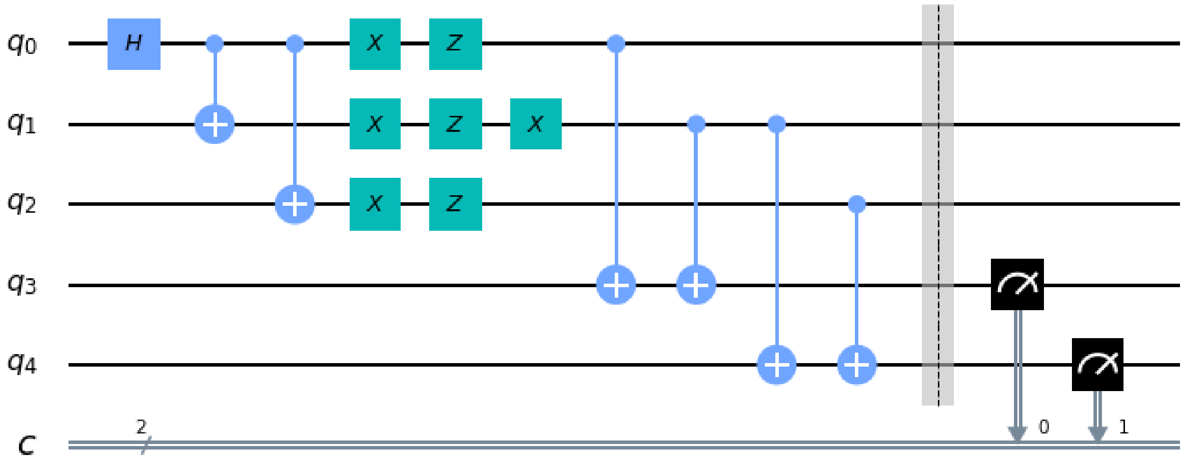
Question :
Write down the state at this point

Three-qubit bit flip code

Our first QEC code: **The three-qubit bit-flip code**

Implementation :

3. Error syndrome (without destroying the qubit superposition, i.e measuring Z_1,Z_2,Z_3)



Question :
Why does this measure $\langle Z_1 Z_2 \rangle$ and $\langle Z_2 Z_3 \rangle$?

Can we correct for phase-flip errors ?

4. Error recovery (easy part and optional)

Lecture 3

Our second QEC code: **The steane code**

Goal: Correct arbitrary single qubit gates

$$|\psi\rangle\otimes|E\rangle \rightarrow \sqrt{1-p_x-p_y-p_z}\,|\psi\rangle\otimes|E\rangle+\sqrt{p_x}X\,|\psi\rangle\otimes|E_x\rangle+\sqrt{p_y}Y\,|\psi\rangle\otimes|E_y\rangle+\sqrt{p_Z}Z\,|\psi\rangle\otimes|E_z\rangle$$

No errors Bit flip Bit+Phase flip Phase flip

For the qubit, this error decomposition **is complete** (proof in terms of Kraus representation of quantum processes)

The Steane code corrects for bit flip, phase flips, and bit+phase flips, thus for arbitrary single qubit errors.

The Steane code

Code world

$$|\psi\rangle = \alpha\,|0\rangle_L + \beta\,|1\rangle_L$$

$$\begin{aligned}|0\rangle_L \;&=\; |0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle \\ &\quad + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle \\ |1\rangle_L \;&=\; X_{1111111}\,|0\rangle_L\,.\end{aligned}$$

→ 1 logical qubits = 7 physical qubits

The Steane code

$$
\begin{aligned}
|0\rangle_L &= |0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle \\
&\quad + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle \\
|1\rangle_L &= X_{1111111}\,|0\rangle_L \,.
\end{aligned}
$$

Syndrome\Error	**0**	**X_1**	**X_2**	**X_3**	**X_4**	**X_5**	**X_6**	**X_7**	**Z_1**	**Z_2**	**Z_3**	**Z_4**	**Z_5**	**Z_6**	**Z_7**	**Y_1**	**…**
$Z_4Z_5Z_6Z_7$	1	1	1	1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	
$Z_2Z_3Z_6Z_7$	1	1	-1	-1	1	1	-1	-1	1	1	1	1	1	1	1	1	
$Z_1Z_3Z_5Z_7$	1	-1	1	-1	1	-1	1	-1	1	1	1	1	1	1	1	-1	
$X_4X_5X_6X_7$	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	1	
$X_2X_3X_6X_7$	1	1	1	1	1	1	1	1	1	-1	-1	1	1	-1	-1	1	
$X_1X_3X_5X_7$	1	1	1	1	1	1	1	1	-1	1	-1	1	-1	1	-1	-1	

Strict Conditions for QEC :

→ (1) The code world is not affected by the syndrome measurement

→ (2) A unique error syndrome per error

Stabilizer formalism

Strict Conditions for QEC :

→ (1) Any logical qubit state is not affected by the syndrome measurement

→ (2) A unique error syndrome per error

These conditions can be easily checked for codes that are written in the stabilizer formalism

Stabilizer code [n,k] (Gottesman and coworkers, 1997)

→ Consider n physical qubits.

→ Consider S a subgroup in the group of Pauli matrices generated by (n-k) commuting elements $g_1, \ldots, g_{n-k}$

→ Then $V_{S,}$ the vector space stabilized by S, is of dimension 2^k, i.e. can encode k logical qubits

→ The set of possible Errors $\{E_k\}$ that can be corrected are such that for any j,k

(1) $E_j^{dag} E_k$ is in S

Or (2) $E_j^{dag} E_k$ anticommutes with one element of S

Stabilizer formalism

Stabilizer code [n,k]

$V_{S,}$ the vector space stabilized by S, is of dimension 2^k, i.e. can encode k logical qubits

The set of possible Errors $E=\{E_k\}$ that can be corrected are such that for any j,k

(1) $E_j^{dag}E_k$ is in S

Or (2) $E_j^{dag}E_k$ anticommutes with one element of S

QEC recipe: The code world is given by two orthonormal state of the vector space V_s

Error Syndromes : Stabilizer measurements

Example **The Bit-flip code is a [3,1] stabilizer code!**

$E=\{I,X_1,X_2,X_3\}$ can be corrected by measuring the stabilizer generators $S=\{Z_1Z_2,Z_2Z_3\}$

The logical states are $|000\rangle$, $|111\rangle$ belong to the vector space V_S

Exercice Prove that the Steane code is a [7,1] stabilizer code.

The challenge of fault-tolerant quantum computing

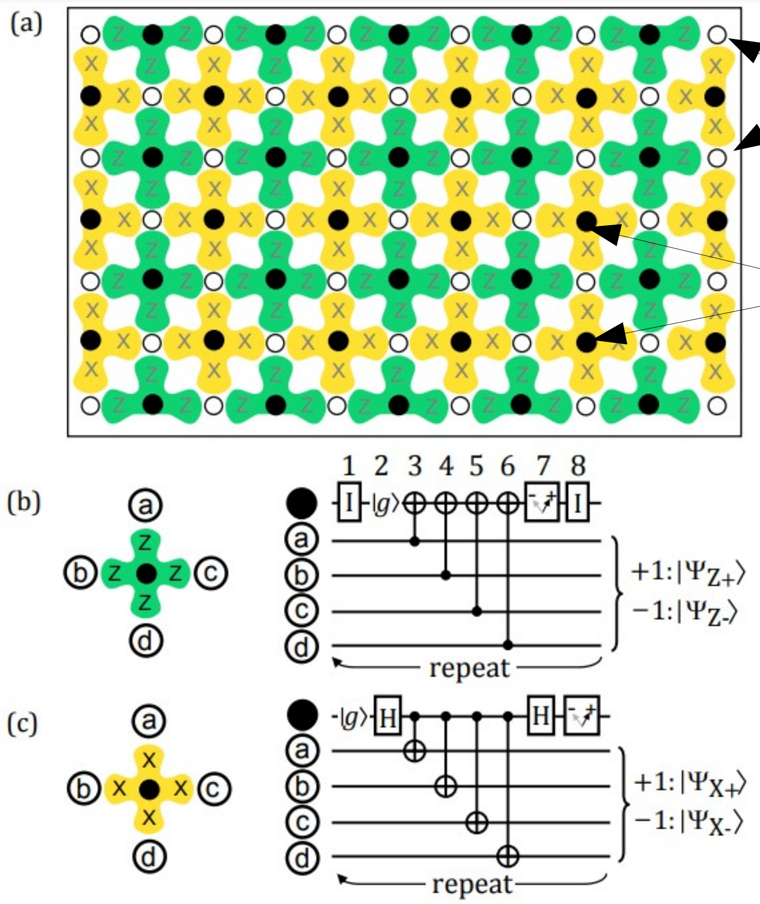
A single QEC code does not protect against **any** error. (ex error X_1X_2 in the bit flip code..)

However, we may concatenate/combine several QEC codes to fight for error propagation provided the error probability per gate is below a certain threshold [Michael Ben-Or and Dorit Aharonov]

Fault-tolerant quantum computing = A significant technological challenge.
Estimation : ~10 000 physical qubits per logical qubit for the surface code

J. Preskill « I've already emphasized repeatedly that it will probably be a long time before we have fault-tolerant quantum computers solving hard problems. »

Towards fault-tolerance with the surface code



Physical qubits

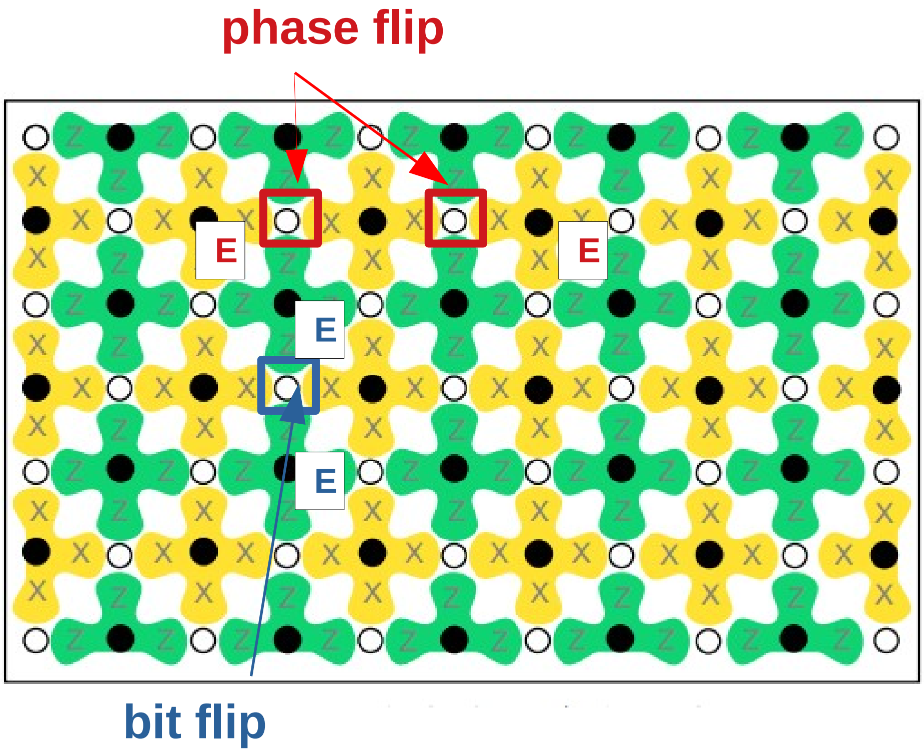
Ancilla qubits → perform stabilizer measurements

All $Z_aZ_bZ_cZ_d$ and $X_aX_bX_cX_d$ (commuting operators)

Ref : https://arxiv.org/pdf/1208.0928.pdf

Towards fault-tolerance with the surface code

Error syndrome = pattern of stabilization results



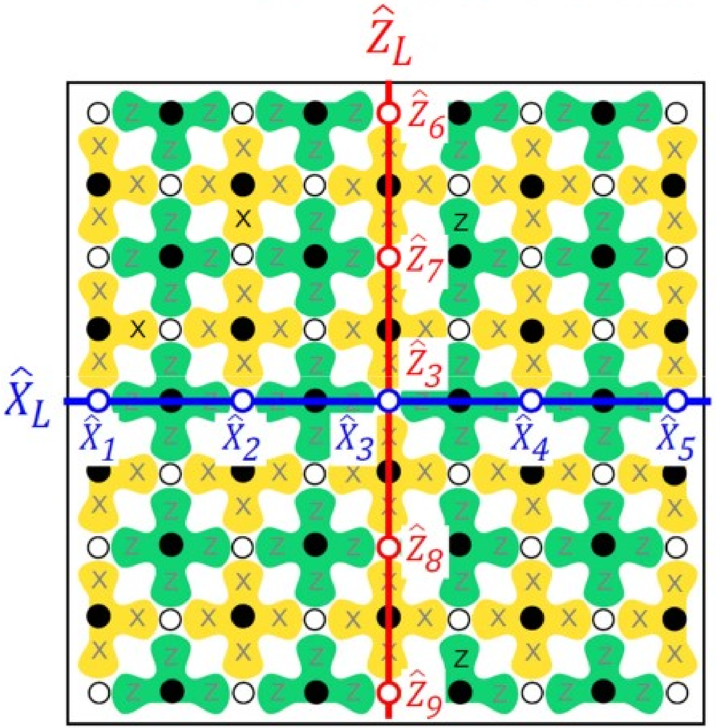
Towards fault-tolerance with the surface code

Initilization and operation

1) Measure all stabilizers
→ Creates a logical $|\psi\rangle$ state

2) Apply logical operations
X_L Z_L (Preserve the code world)

3) Check for errors by stabilizer measurements

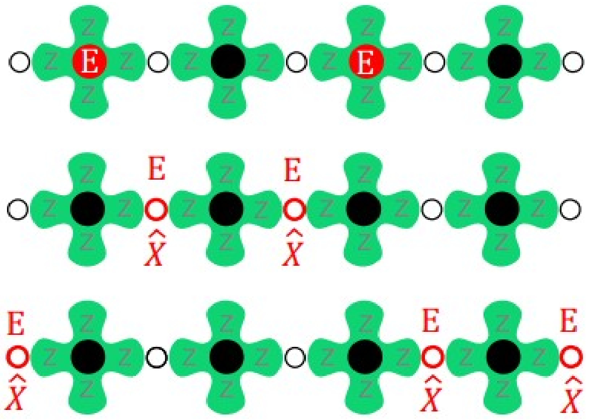


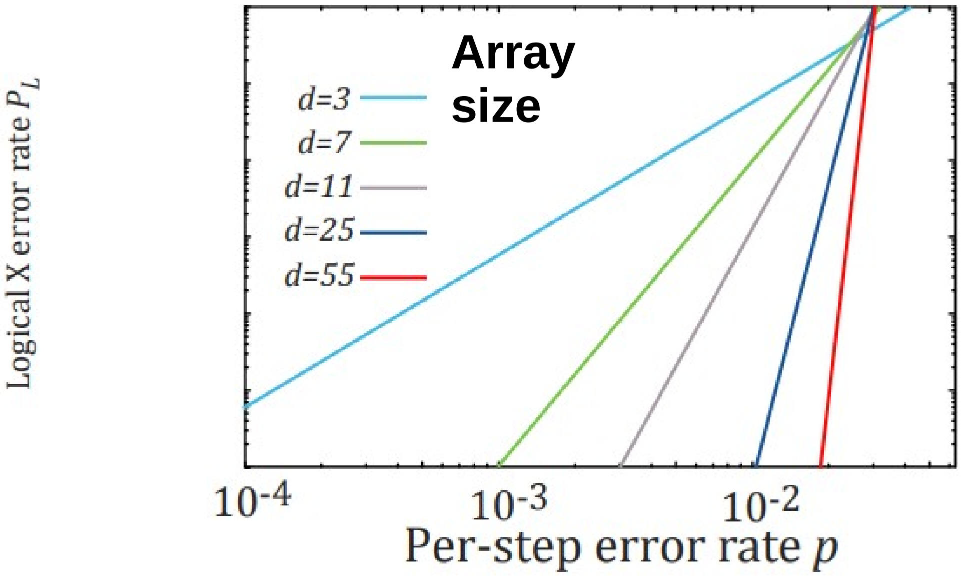
Towards fault-tolerance with the surface code

Robustness of the logical qubit

QEC fails when the numbers of errors~system size d/2

Example : One error syndrome for two possible errors

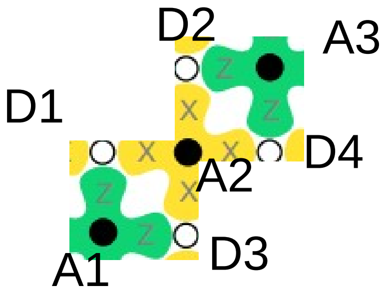


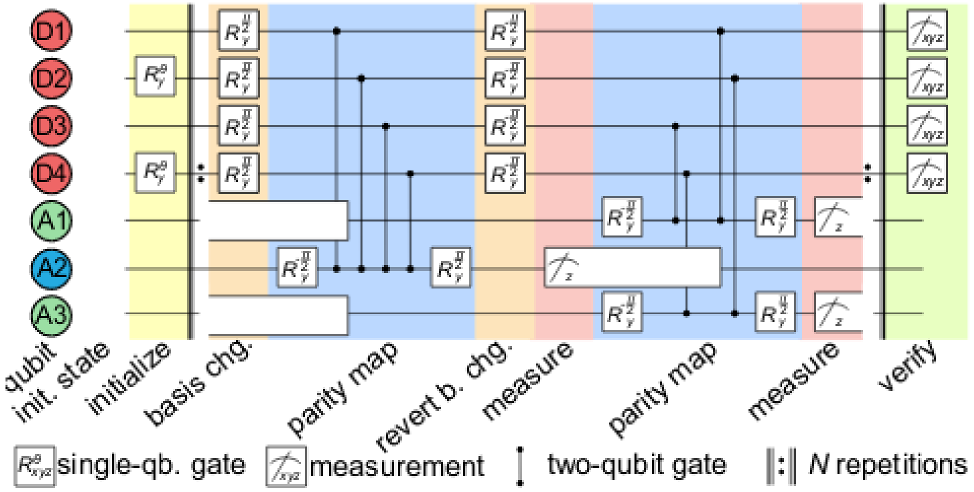


Fault-tolerance can be achieved by increasing the array size

Towards fault-tolerance with the surface code

Experimental implementation of a minimal surface code *https://arxiv.org/pdf/1912.09410.pdf*





Stabilizer : $Z_{D1}Z_{D3}$, $Z_{D2}Z_{D4}$, and $X_{D1}X_{D2}X_{D3}X_{D4}$

$$|0\rangle_L = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$$

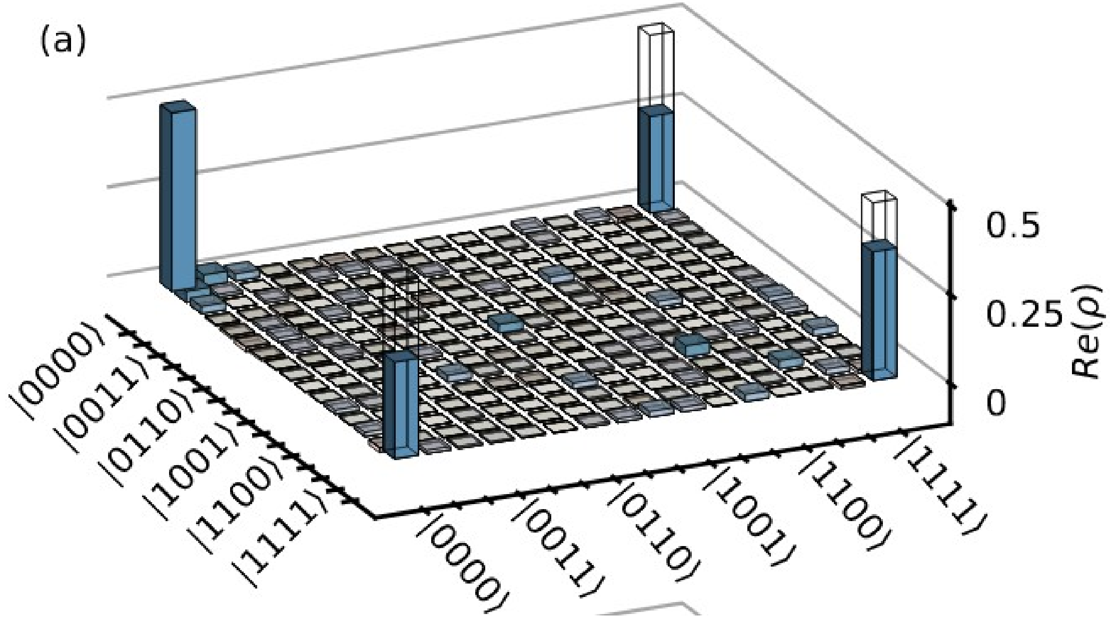
$$|1\rangle_L = \frac{1}{\sqrt{2}}(|0101\rangle + |1010\rangle)$$

Towards fault-tolerance with the surface code

Preparation of the 0_L logical state

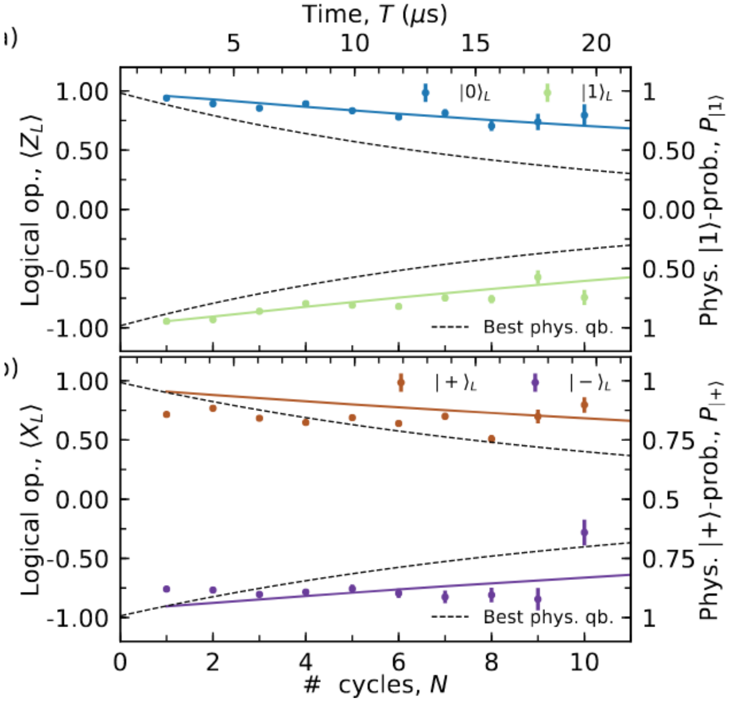
→ start from 0^4

→ measure all ancillas in 0

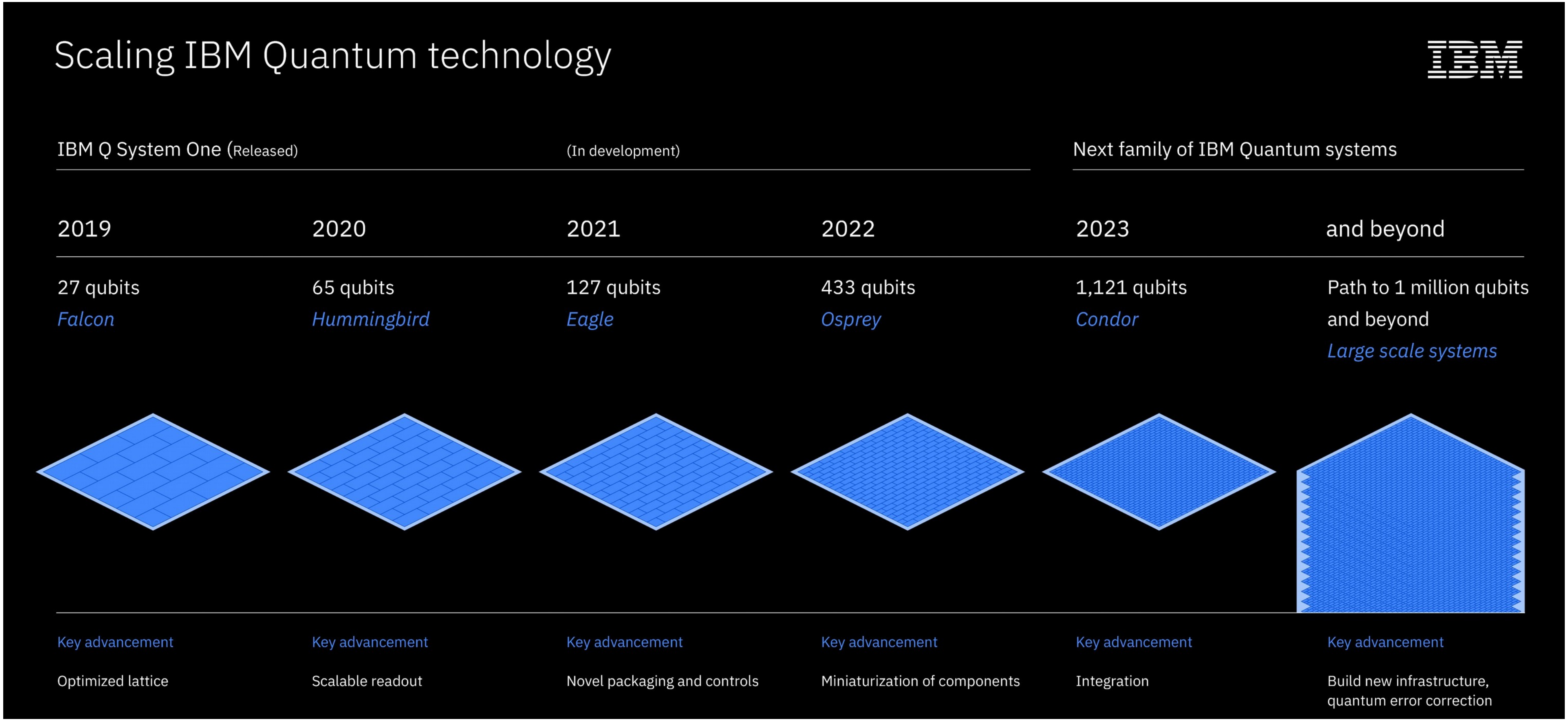


Repeted error correction

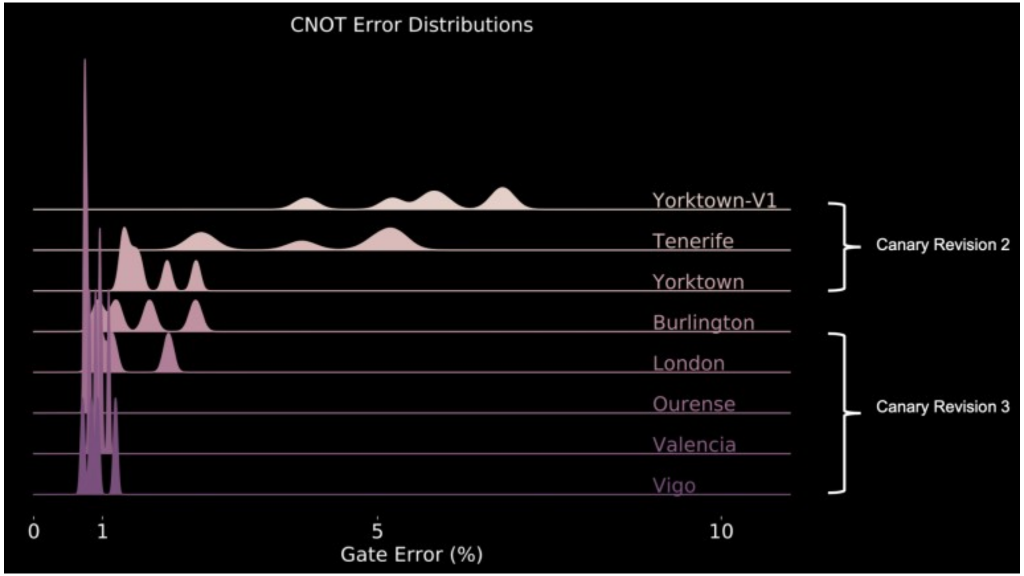
(postprocessing on the no-error signal)

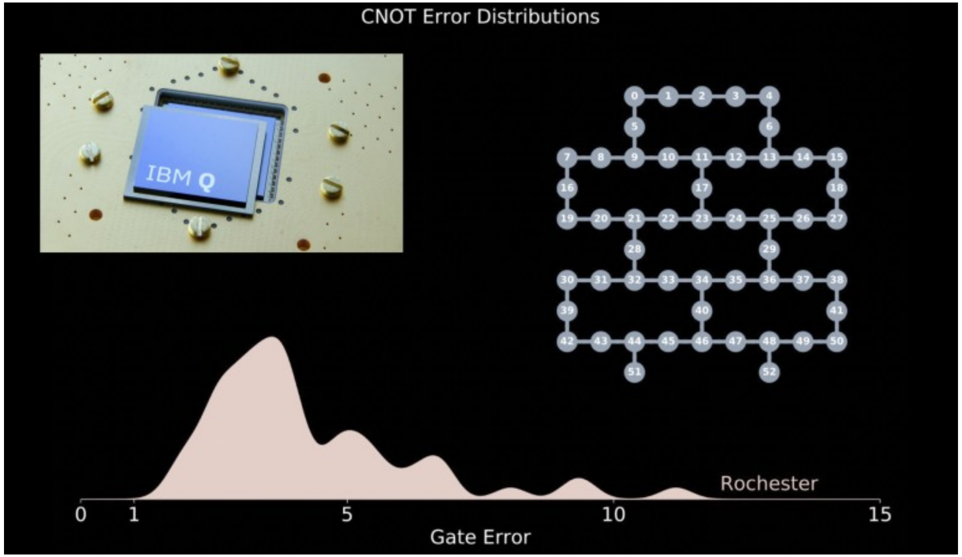


Consequence : we need more physical qubits



Consequence : we need more physical qubits





Lecture 3 : Summary

QEC is a hot topic in modern quantum computing and **a key challenge** for quantum technologies

Fault-tolerant quantum computing requires many physical qubit per logical qubit

Surface codes implement QEC as 2D spin lattice models, which we also encounter in quantum simulation : Lecture 4