Quantum algorithms

Lecture 1: Introduction and quantum circuits

Benoît Vermersch

March 1 2022

LPMMC Grenoble

Outline

From classical to quantum computers

Lecture 1: Quantum circuits

- Single qubit states/gates
- Two qubit gates and universal quantum computing

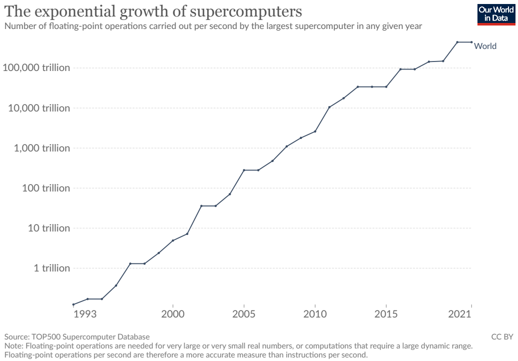
Outline

From classical to quantum computers

Lecture 1: Quantum circuits

- Single qubit states/gates
- Two qubit gates and universal quantum computing

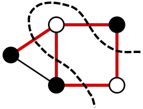
The exponential growth of supercomputers



- Source Wikipedia
- Frontier (US) 10^{18} FLOPS - 600 million USD.
- Why do we need so much computation power?

Some tough problems for classical computers

- Integer factorization: $N = ab$.
- Search algorithms: $f(x = w) = 1$, $f(x \neq w) = 0$. Find w.
- Optimization problems/Machine Learning: Given a cost function $f(x)$, find x that maximizes $f(x)$



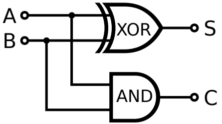
- Quantum problems: quantum chemistry, superconductivity, etc $\rightarrow$ solve large-scale Schrödinger equation

Basic concepts of computer science

- A useful model for computers: circuits of logical gates acting on binary numbers.
- AND gate: $a' = ab \rightarrow$ Truth table:

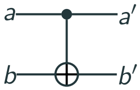
a	b	a'
0	0	0
0	1	0
1	0	0
1	1	1

- OR gate: $a' = a + b \rightarrow$ Truth table: ...
- XOR gate: $a' = a \oplus b \rightarrow$ Truth table: ...
- Combining gates, we obtain *logical circuits*, eg Half-adder

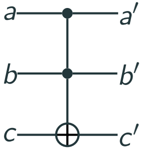


Basic concepts of computer science

- Reversible circuits: $(a', b') = f(a, b)$, with f invertible.
- Motivation: Irreversible processes increase entropy production during computation (Landauer's principle).
- Reversible XOR gate (known as CNOT in quantum computing)
  - $a' = a,\ b' = a \oplus b$
  - Remark: the second ‘target’ bit is flipped iff the first ’control’ bit is activated $(a = 1)$.
  - Useful notations for later



- Universality in reversible computing: Can I write using a finite set of gates an arbitary reversible circuit f, $(a'_1, \ldots, a'_n) = f(a_1, \ldots, a_n)$?
- The Toffoli gate is universal (see TD1)

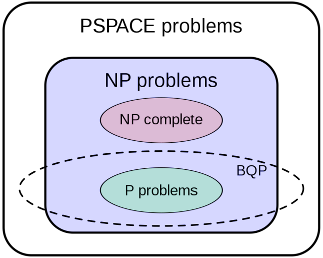


Complexity classes

Complexity: scaling of resources to solve a decision problem (yes/no answer) with a classical computer (rigorously, for a deterministic Turing machine)

P	Solved in polynomial time, i.e.,
	the required number of operations is a polynomial of a problem size
NP	A yes answer verified in polynomial time
PSPACE	Solved with polynomial size (i.e number of constituents, bits)
EXPTIME	Solved in exponential time
NP-HARD	Every problem in NP can be transformed
	into this problem in polynomial time
NP-COMPLETE	A problem that is both NP and NP-HARD

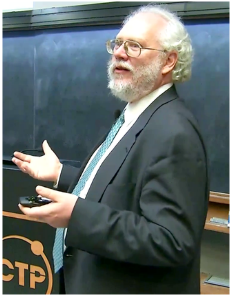
Relations between complexity classes



- Conjecture P$\neq$NP
- BQP (Bounded error quantum polynomial time) is the class associated with quantum computers
- We believe that P$\neq$BQP, ie quantum computers may be useful!

Example: Integer factorization

- Factorization decision problem (F): can a given n-bit number be factorized?
- No polynomial time algorithm known: We do not know if F is in P
- Solutions can be checked efficiently: F is in NP
- Shor's algorithm (1995): F is in BQP



- Quantum computers may be able to tackle problems that are hard for classical computers!

From classical to quantum computers in the quantum circuit model

Classical computers use classical bits

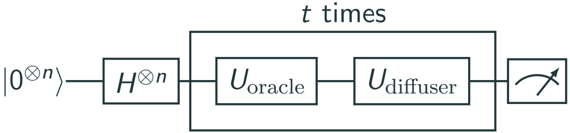
- One classical bit:
  $\rightarrow$ state in 0 or 1
- n classical bits: ...
  $\rightarrow$ 2^n possibilities for the state
  $(00\ldots00,\ 00\ldots01, \text{etc})$

Quantum computers use qubits

- One qubit:
  $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$
- n qubits: ...
  $|\psi\rangle = c_{0\ldots0}\,|0\ldots0\rangle + c_{0\ldots1}\,|0\ldots1\rangle + \ldots.$
  The quantum state can be simulatenously in all the 2^n classical states.

Why we may expect quantum speedup with quantum parallelism

- Unstructured search on a space of 2^n bitstrings: We look for x, such that $f(x = x_1, \ldots, x_n) = 1$.
- Optimal classical algorithm: Random testing, with time complexity $O(2^n)$
- Optimal quantum algorithm: Grover's algorithm (Lecture 2)

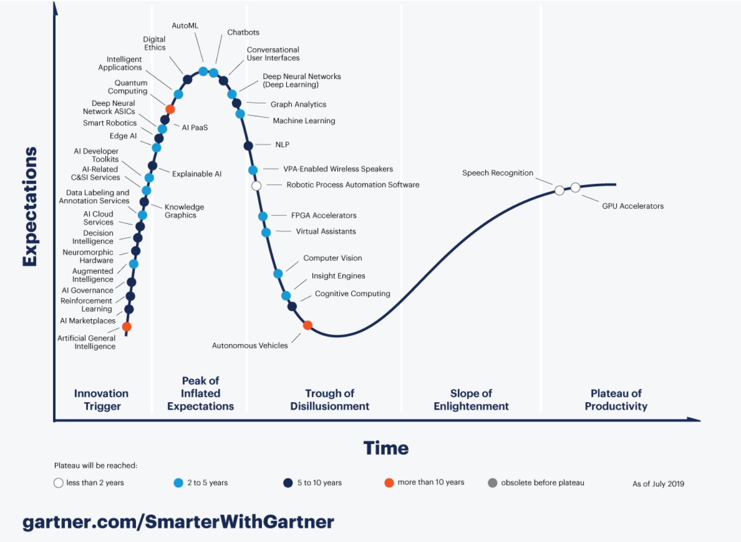


  - The first quantum gate creates a uniform quantum superposition of all bitstring states
  - Complexity $O(\sqrt{2^n})$: polynomial improvement (but still exponential scaling).
- Note: Quantum parallelism does not guarantee quantum speedup for solving *anything*: One needs to extract relevant classical information from the quantum superposition state.

Quantum Computing timeline

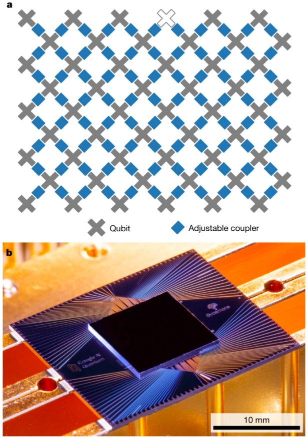
- 1980-1981: Concepts of quantum computers (Benioff-Manin-Feynman)
- 1985: Deutsch's universal quantum computer (Lecture 1)
- 1994: Shor's factoring algorithm (Lecture 3)
- 1995: Shor proposes quantum error correction (Lecture 4)
- 1995: First realization of a 2 qubit gate with trapped ions (Wineland)
- 1996: Grover's algorithm (Lecture 2)
- 2001: 15 is factorized with Shor's algorithm at IBM
- 2008: D-Wave Systems propose the first commercially available quantum computer (Lecture 5)
- 2019: Quantum supremacy claim by Google with 53 qubits (Lecture 6)
- 2021: IBM quantum eagle (127 qubits)

The quantum ‘hype’



Physical realizations: quantum hardware

- Superconducting qubits (Google, IBM, Rigetti, Grenoble, . . .)
- Trapped ions (Innsbruck, Duke university, Boulder NIST, IonQ, . . .).
- Rydberg atoms (Palaiseau, Pasqal, Harvard, . . .)
- Electron spins (Delft, Microsoft, Grenoble, . . .)

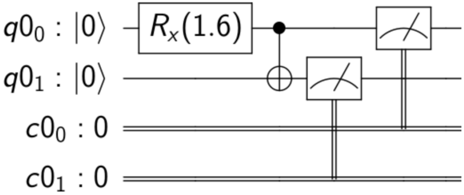


IBMQ Practicals

- Organization: Julien Renard and BV.
- We will use via a cloud interface ‘small’ IBM quantum computers to illustrate the lectures.
- We will use the Qiskit Python library to parametrize simulate quantum circuits and interface with the quantum machines.

In [7]:
```
from qiskit import QuantumRegister, ClassicalRegister, QuantumCircui
from qiskit.tools.visualization import circuit_drawer
import numpy as np

qr = QuantumRegister(2)
cr = ClassicalRegister(2)
qp = QuantumCircuit(qr,cr)

qp.rx( np.pi/2,qr[0])
qp.cx(qr[0],qr[1])

qp.measure(qr,cr)

circuit_drawer(qp)
```

Out[7]:



IBMQ Practicals: before the first practical

- Connect to Qiskit.org
- Install the Python Qiskit library on your laptop (eg via Anaconda)
- Test to import the library
- Create an IBMQ quantum experience account and get an API token

Organization

- Lecture 1: Quantum circuits
- Lecture 2: Quantum algorithms (1)
- Lecture 3: Quantum algorithms (2)
- Lecture 4: Quantum error correction
- Lecture 5: Quantum simulation/quantum optimization
- Lecture 6: Bonus topics

Resources

- Nielsen and Chuang, Quantum Computation and Quantum Information
- J. Preskill's quantum information lectures, `http://theory.caltech.edu/~preskill/`
- Lectures slides/Exercices/Schedule on Moodle Quantum Algorithms
- Groups/Schedule Moodle IBMQ

Outline

From classical to quantum computers

Lecture 1: Quantum circuits

- Single qubit states/gates
- Two qubit gates and universal quantum computing

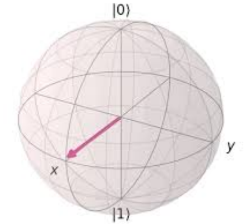
Single qubit states

- For this course, the qubit can be thought as the elementary building block of a quantum computer.
- A qubit is a two-level quantum system

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{1}$$

- Sometime it is instructive to represent a qubit as a vector on the Bloch sphere

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|1\rangle \tag{2}$$



Single qubit quantum circuit

- In single qubit quantum circuits, the qubit states evolves as a function of time, by successive applications of single qubit gates

$$|\psi\rangle \;-\!\!\!-\; \boxed{U_1} \;-\!\!\!-\!\!\!-\!\!\!-\; \boxed{U_2} \;-\!\!\!-\!\!\!-\!\!\!-\; \boxed{U_3} \;-\!\!\!-$$

- After the first gate, we obtain

$$|\psi_1\rangle = U_1 |\psi\rangle \,, \tag{3}$$

  with $U_1 = e^{-iH_1 t}$ is a unitary 2×2 matrix (a rotation on the Bloch sphere)

- Can you write the final state of the circuit as a function of U_1, U_2, U_3?

Important single qubit gates

- Note: It is important to get used to calculate the states of quantum circuits both using the matrix and the bra-ket notations

X-gate $\quad -\boxed{X}- \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad X = |0\rangle\langle 1| + |1\rangle\langle 0|$

Z-gate $\quad -\boxed{Z}- \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad Z = |0\rangle\langle 0| - |1\rangle\langle 1|$

Y-gate $\quad -\boxed{Y}- \quad Y = iXZ$

Hadamard-gate $\quad -\boxed{H}- \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

T-gate $\quad -\boxed{T}- \quad T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$

and others..

- Can you generate the Hadamard gate with a circuit with X and Z gates? why?

Multi-qubit circuit structure

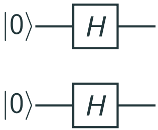
- A quantum circuit naturally extends to $n \geq 1$ qubits
- The wave-function is written in a tensor product space of dimension 2^n

$$|\psi\rangle = \sum_{x_1=0}^{1} \sum_{x_2=0}^{1} \cdots \sum_{x_n=0}^{1} c_{x_1,x_2\ldots,x_n} |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle = \begin{pmatrix} c_{0,0,\ldots 0,0} \\ c_{0,0\ldots 0,1} \\ \ldots \\ c_{1,1\ldots 1,1} \end{pmatrix} \tag{4}$$

- Equivalence between notations :
  $|0,1,1\rangle = |011\rangle = |0\rangle \otimes |1\rangle \otimes |1\rangle$, $|0\rangle \otimes |0\rangle = |0\rangle^{\otimes 2}$

Multi-qubit circuit structure

- Let us calculate the state after the following two-qubit circuit?



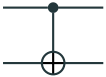
- Playing with bra-kets and tensor products

$$|\psi\rangle = (H \otimes H)(|0\rangle \otimes |0\rangle) = (H|0\rangle) \otimes (H|0\rangle) = \dots \tag{5}$$

- I can also first write H in bra-ket notations then write $H \otimes H$ in bra-ket notations, or in matrix form, etc, but it's more tedious.
- Is this state entangled?

Introducing two-qubit gates

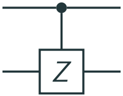
- Two qubit gates act non-trivially on two qubits
- Example CNOT gate



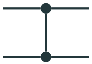
$$CNOT = |0\rangle\langle 0| \otimes 1 + |1\rangle\langle 1| \otimes X \tag{6}$$

  Is this actually unitary?

- Example Controlled-Z gate



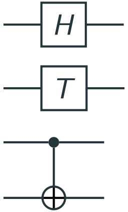
or



$$CZ = |0\rangle\langle 0| \otimes 1 + |1\rangle\langle 1| \otimes Z \tag{7}$$

The universal quantum computer

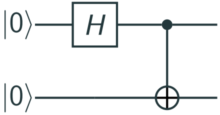
- Deutsch 1985: There exist universal set of gates that can be used to generate any quantum circuit U acting on n qubits.
- Note: The question of how many gates you need is non-trivial (quantum computational complexity)
- The following gate set is universal



- It is usually a good idea to use a larger gate set to simplify the circuit compilation.
- Write a circuit to create a Bell State, a three qubit GHZ state.

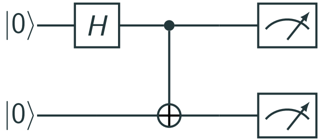
Bell state circuit

Circuit to prepare the Bell state



The measurement

- Every quantum circuits ends up with a measurement

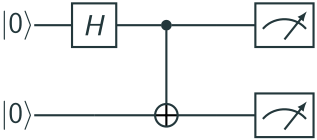


that produces random output $x_1, \ldots, x_n$ based on Born probabilities.

$$P(x_1, \ldots, x_n) = |\langle x_1, \ldots, x_n|\psi\rangle|^2 \tag{8}$$

- The post-select states $|x_1, \ldots, x_n\rangle$ is obtained by application of the projection operator $P = |x_1\rangle\langle x_1| \ldots |x_n\rangle\langle x_n|$.

The measurement problem



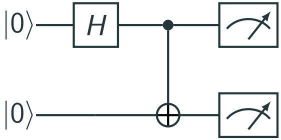
- I need to run the experiment M times, accessing $m = 1, \ldots, M$ bitstrings $x_m = (x_{m,1}, \ldots, x_{m,n})$ to access meaningful information.

$$| \langle 00|\psi\rangle |^2 = \lim_{M\to\infty} \sum_{m=1}^{M} \frac{\delta_{x_m,(0,0)}}{M} \tag{9}$$

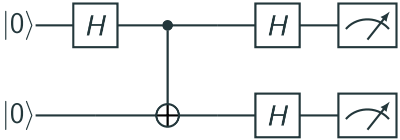
- The measurement problem is a crucial part in the design of quantum algorithms.

Some common measurement circuits

- Z Basis measurements: Gives access to Born probabilities and arbitrary expectation values involving only Z operators.



- One can also apply single qubit gates before the measurement, eg X Basis measurements:

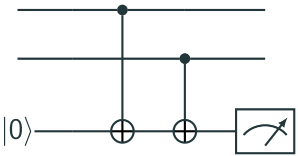


- The key identity is $Z = HXH$ (see $TD1$)

Projection aspects and ancilla based measurements

- What happens if we do not measure entirely the system?
- Example: This measurement circuit takes a 2-qubit state $|\psi\rangle$, build a 3-qubit state $|\psi'\rangle$

  and delivers one measurement outcome $\nu = 0, 1$.



- This is a (von Neumann) quantum measurement with projection operators $P_\nu = 1 \otimes 1 \otimes |\nu\rangle\langle\nu|$, $\nu = 0, 1$, which project the system into

$$P_\nu |\psi'\rangle \quad \text{with probability} \quad \langle\psi'|P_\nu|\psi'\rangle \tag{10}$$

- This type of measurements will be essential for error correction (Lecture 4).

Information: Quantum algorithms

- Contact franck.balestro@neel.cnrs.fr if you want to register to the class (credits/exam/etc)
- Contact me benoit.vermersch@lpmmc.cnrs.fr for the Moodle registration and pedagogical info
- Check the Moodle frequently

Information: IBM Quantum Practicals

- Contact franck.balestro@neel.cnrs.fr if you want to register to the class (credits/exam/etc)
- Contact me benoit.vermersch@lpmmc.cnrs.fr for the Moodle registration and pedagogical info
- Find out in which group you are, and the schedule
- Install Qiskit, Register to IBM Quantum
- You will receive an invitation to join the Educators Program (gives priority in the queuing system)