Introduction to quantum computing

Lecture 1: From classical to quantum computers

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LPMMC Grenoble

Outline

From classical to quantum computers

Lecture 1: Quantum circuits

- Single qubit states/gates
- Two qubit gates and universal quantum computing

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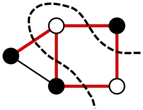
- Two qubit gates and universal quantum computing

Some tough problems for classical computers

- Integer factorization: $N = ab$.
- Search algorithms: $f(x = w) = 1$, $f(x \neq w) = 0$. Find w.
- Optimization problems: Given a cost function $f(x)$, find x that maximizes $f(x)$



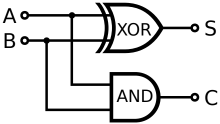
- Quantum problems: quantum chemistry, superconductivity, etc $\rightarrow$ solve large-scale Schrödinger equation

Basic concepts of computer science

- A useful model for computers: circuits of logical gates acting on binary numbers.
- AND gate: $a' = ab \rightarrow$ Truth table:

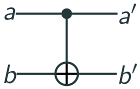
a	b	a'
0	0	0
0	1	0
1	0	0
1	1	1

- OR gate: $a' = a + b \rightarrow$ Truth table: ...
- XOR gate: $a' = a \oplus b \rightarrow$ Truth table: ...
- Combining gates, we obtain *logical circuits*, eg Half-adder

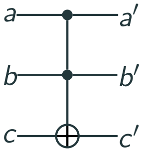


Basic concepts of computer science

- Reversible circuits: $(a', b') = f(a, b)$, with f invertible.
- Reversible XOR gate (known as CNOT in quantum computing)
  - $a' = a$, $b' = a \oplus b$
  - Remark: the second ‘target’ bit is flipped iff the first ’control’ bit is activated ($a = 1$).
  - Useful notations for later



- Universality in reversible computing: Can I write using a finite set of gates an arbitrary reversible circuit f, $(a'_1, \ldots, a'_n) = f(a_1, \ldots, a_n)$?
- The Toffoli gate is universal



Complexity classes

Complexity: scaling of resources to solve a decision problem (yes/no answer) with a classical computer (rigorously, for a deterministic Turing machine)

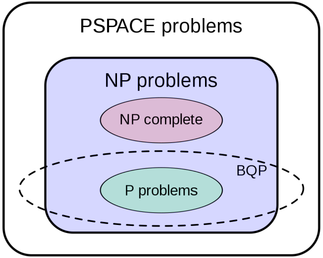
P	Solved in polynomial time, i.e., the required number of operations is a polynomial of a problem size
NP	A yes answer verified in polynomial time
PSPACE	Solved with polynomial size (i.e number of constituents, bits)
EXPTIME	Solved in exponential time
NP-HARD	Every problem in NP can be transformed into this problem in polynomial time
NP-COMPLETE	A problem that is both NP and NP-HARD

Example of a NP-complete problem: 3-SAT

$$
\begin{aligned}
&(\overline{a} \vee m \vee u) \wedge (a \vee n \vee u) \wedge (\overline{a} \vee r \vee x) \wedge (\overline{c} \vee \overline{e} \vee s) \\
\wedge &(c \vee \overline{m} \vee \overline{w}) \wedge (\overline{c} \vee p \vee x) \wedge (c \vee q \vee s) \wedge (e \vee p \vee s) \\
\wedge &(e \vee q \vee \overline{y}) \wedge (e \vee r \vee y) \wedge (\overline{e} \vee r \vee z) \wedge (\overline{g} \vee r \vee x) \\
\wedge &(g \vee v \vee \overline{y}) \wedge (m \vee \overline{n} \vee u) \wedge (m \vee \overline{o} \vee \overline{u}) \wedge (m \vee o \vee v) \\
\wedge &(\overline{m} \vee \overline{q} \vee s) \wedge (\overline{m} \vee \overline{r} \vee \overline{s}) \wedge (m \vee \overline{u} \vee \overline{v}) \wedge (\overline{m} \vee x \vee \overline{z}) \\
\wedge &(\overline{n} \vee r \vee \overline{y}) \wedge (o \vee r \vee \overline{w}) \wedge (\overline{p} \vee q \vee s) \wedge (r \vee \overline{w} \vee \overline{x}) \\
\wedge &(r \vee w \vee \overline{y}) \wedge (r \vee w \vee \overline{z})
\end{aligned}
$$

source:Wikipedia

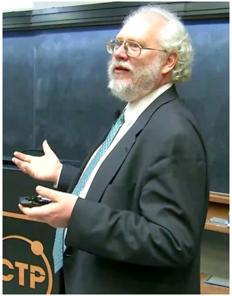
Relations between complexity classes



- Conjecture P$\neq$NP
- BQP (Bounded error quantum polynomial time) is the class associated with quantum computers
- We believe that P$\neq$BQP, ie quantum computers may be useful!

Example: Integer factorization

- Factorization decision problem (F): can a given *n*-bit number be factorized?
- No polynomial time algorithm known: We do not know if F is in P
- Solutions can be checked efficiently: F is in NP
- Shor's algorithm (1995): F is in BQP



- Quantum computers may be able to tackle problems that are hard for classical computers!

From classical to quantum computers in the quantum circuit model

Classical computers use classical bits

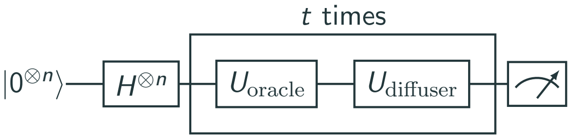
- One classical bit: 
  $\rightarrow$ state in 0 or 1
- n classical bits: 
  $\rightarrow 2^n$ possibilities for the state (00 . . . 00, 00 . . . 01,etc)

Quantum computers use qubits

- One qubit: 
  $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$
- n qubits: 
  $|\psi\rangle = c_{0\ldots0}|0\ldots0\rangle + c_{0\ldots1}|0\ldots1\rangle + \ldots.$
  The quantum state can be simulatenously in all the 2^n classical states.

Why we may expect quantum speedup with quantum parallelism

- Unstructured search on a space of 2^n bitstrings: We look for x, such that $f(x = x_1, \ldots, x_n) = 1$.
- Optimal classical algorithm: Random testing, with time complexity $O(2^n)$
- Optimal quantum algorithm: Grover's algorithm (Lecture 2)



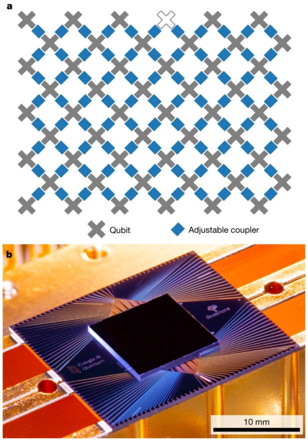
  - The first quantum gate creates a uniform quantum superposition of all bitstring states
  - Complexity $O(\sqrt{2^n})$: polynomial improvement (but still exponential scaling).
- Note: Quantum parallelism does not guarantee quantum speedup for solving *anything*: One needs to extract relevant classical information from the quantum superposition state.

Quantum Computing timeline

- 1980-1981: Concepts of quantum computers (Benioff-Manin-Feynman)
- 1985: Deutsch's universal quantum computer
- 1994: Shor's factoring algorithm
- 1995: Shor proposes quantum error correction
- 1995: First realization of a 2 qubit gate with trapped ions (Wineland)
- 1996: Grover's algorithm
- 2001: 15 is factorized with Shor's algorithm
- 2008: D-Wave Systems propose the first commercially available "quantum computer"
- 2019: Quantum supremacy claim by Google with 53 qubits
- 2021: IBM quantum eagle (127 qubits)
- 2023: First "large" "low-depth" logical quantum processor (Harvard & QuEra)

Physical realizations: quantum hardware

- Superconducting qubits (Google, IBM, Rigetti, Grenoble, . . .)
- Trapped ions (Innsbruck, Duke university, Boulder NIST, IonQ, . . .).
- Rydberg atoms (Palaiseau, Pasqal, Harvard, . . .)
- Electron spins (Delft, Microsoft, Grenoble, . . .)



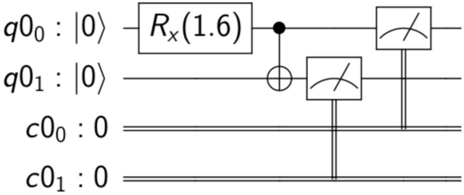
IBMQ Practicals

- We will use via a cloud interface 'small' IBM quantum computers to illustrate the lectures.
- We will use the Qiskit Python library to parametrize simulate quantum circuits and interface with the quantum machines.
- Before the class, connect to https://github.com/bvermersch/bvermersch.github.io/blob/master/Teaching/QuantumPractical.ipynb for install instructions.

In [7]:

```
from qiskit import QuantumRegister, ClassicalRegister, QuantumCircui
from qiskit.tools.visualization import circuit_drawer
import numpy as np

qr = QuantumRegister(2)
cr = ClassicalRegister(2)
qp = QuantumCircuit(qr,cr)

qp.rx( np.pi/2,qr[0])
qp.cx(qr[0],qr[1])

qp.measure(qr,cr)

circuit_drawer(qp)
```

Out[7]:



Resources

- Nielsen and Chuang, Quantum Computation and Quantum Information
- J. Preskill's quantum information lectures, `http://theory.caltech.edu/~preskill/`
- Scott's Aaronson lectures.
- Lectures slides on https://bvermersch.github.io/

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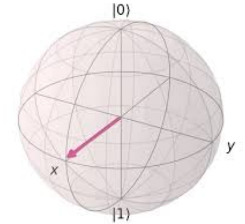
Single qubit states

- For this course, the qubit can be thought as the elementary building block of a quantum computer.
- A qubit is a two-level quantum system

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{1}$$

- Sometime it is instructive to represent a qubit as a vector on the Bloch sphere

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|1\rangle \tag{2}$$



Single qubit quantum circuit

- In single qubit quantum circuits, the qubit states evolves as a function of time, by successive applications of single qubit gates

- After the first gate, we obtain

$$|\psi_1\rangle = U_1 |\psi\rangle \, , \tag{3}$$

with $U_1 = e^{-iH_1 t}$ is a unitary 2×2 matrix (a rotation on the Bloch sphere)

- Can you write the final state of the circuit as a function of U_1, U_2, U_3?

Important single qubit gates

- Note: It is important to get used to calculate the states of quantum circuits both using the matrix and the bra-ket notations

X-gate 

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad X = |0\rangle\langle 1| + |1\rangle\langle 0|$$

Z-gate 

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad Z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

Y-gate 

$$Y = iXZ$$

Hadamard-gate 

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

T-gate 

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

and others..

- Can you generate the Hadamard gate with a circuit with X and Z gates? why?

Multi-qubit circuit structure

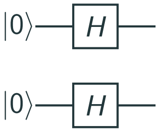
- A quantum circuit naturally extends to $n \geq 1$ qubits
- The wave-function is written in a tensor product space of dimension 2^n

$$|\psi\rangle = \sum_{x_1=0}^{1} \sum_{x_2=0}^{1} \cdots \sum_{x_n=0}^{1} c_{x_1,x_2\ldots,x_n} |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle = \begin{pmatrix} c_{0,0,\ldots 0,0} \\ c_{0,0\ldots 0,1} \\ \ldots \\ c_{1,1\ldots 1,1} \end{pmatrix} \quad (4)$$

- Equivalence between notations :
  $|0,1,1\rangle = |011\rangle = |0\rangle \otimes |1\rangle \otimes |1\rangle$, $|0\rangle \otimes |0\rangle = |0\rangle^{\otimes 2}$

Multi-qubit circuit structure

- Let us calculate the state after the following two-qubit circuit?



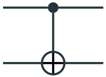
- Playing with bra-kets and tensor products

$$|\psi\rangle = (H \otimes H)(|0\rangle \otimes |0\rangle) = (H\,|0\rangle) \otimes (H\,|0\rangle) = \dots \tag{5}$$

- I can also first write H in bra-ket notations then write $H \otimes H$ in bra-ket notations, or in matrix form, etc, but it's more tedious.
- Is this state entangled?

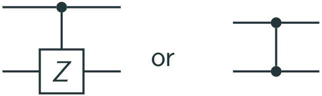
Introducing two-qubit gates

- Two qubit gates act non-trivially on two qubits
- Example CNOT gate



$$CNOT = |0\rangle \langle 0| \otimes 1 + |1\rangle \langle 1| \otimes X \tag{6}$$

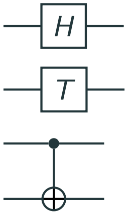
- Example Controlled-Z gate



$$CZ = |0\rangle \langle 0| \otimes 1 + |1\rangle \langle 1| \otimes Z \tag{7}$$

The universal quantum computer

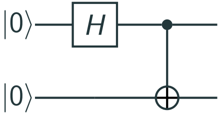
- Deutsch 1985: There exist universal set of gates that can be used to generate any quantum circuit U acting on n qubits.
- Note: The question of how many gates you need is non-trivial (quantum computational complexity)
- The following gate set is universal



- It is usually a good idea to use a larger gate set to simplify the circuit compilation.
- Write a circuit to create a Bell State, a three qubit GHZ state.

Bell state circuit

Circuit to prepare the Bell state



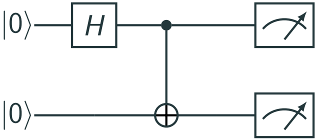
The measurement problem

- Reminer on von Neumann measurements in quantum mechanics
- Let us define a list of orthogonal projectors $\{P_a\}$, with $\sum_a P_a = 1$
- We define a measurement of a as the physical operation

$$P_a |\psi\rangle / || P_a |\psi\rangle || \tag{8}$$

- And we postulate that this happens with probability $p_a = \langle\psi|P_a|\psi\rangle$.

Computational bases measurements



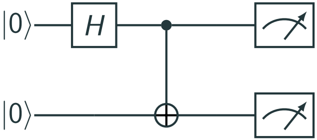
- This is the standard way of measuring in quantum computers

$$P_x = |x\rangle\langle x| \tag{9}$$

with $x = x_1, \ldots, x_n$ and

$$p(x) = |\langle\psi|x\rangle|^2 \tag{10}$$

The measurement problem



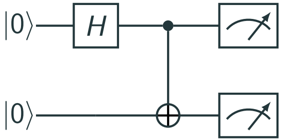
- I need to run the experiment M times, accessing $m = 1, \ldots, M$ bitstrings $x_m = (x_{m,1}, \ldots, x_{m,n})$ to access meaningful information.

$$| \langle 00|\psi\rangle |^2 = \lim_{M\to\infty} \sum_{m=1}^{M} \frac{\delta_{x_m,(0,0)}}{M} \tag{11}$$

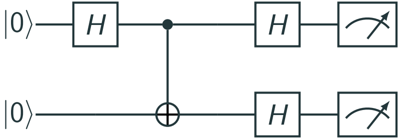
- The measurement problem is a crucial part in the design of quantum algorithms.

Some common measurement circuits

- Z Basis measurements: Gives access to Born probabilities and arbitrary expectation values involving only Z operators.



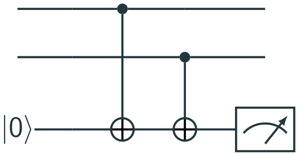
- One can also apply single qubit gates before the measurement, eg X Basis measurements:



- The key identity is $Z = HXH$.

Projection aspects and ancilla based measurements

- What happens if we do not measure entirely the system?
- Example: This measurement circuit takes a 2-qubit state $|\psi\rangle$, build a 3-qubit state $|\psi'\rangle$ and delivers one measurement outcome $\nu = 0, 1$.



- This is a measurement with projection operators $P_\nu = 1 \otimes 1 \otimes |\nu\rangle\langle\nu|$, $\nu = 0, 1$, which project the system into

$$P_\nu |\psi'\rangle \quad \text{with probability} \quad \langle\psi'|P_\nu|\psi'\rangle \tag{12}$$