Quantum Algorithms 2021/2022: Exercices 2

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1 Implementation of Grover's diffuser operator

Our goal is to design a quantum circuit for $U_\psi = 2\left|\psi\right\rangle\left\langle\psi\right| - 1$.

1. $U_1 = H^{\otimes n}$
2. $U_1^2 = (H^{\otimes n})(H^{\otimes n} = (H^2)^{\otimes n} = 1$.
3. $U_\psi = 2U_1\left|0\right\rangle^{\otimes n}\left\langle 0\right|^{\otimes n} U_1 - U_1^2 = U_1(2\left|0\right\rangle^{\otimes n}\left\langle 0\right|^{\otimes n} - 1)U_1$
4. $U_2 = 2\left|0\right\rangle^{\otimes n}\left\langle 0\right|^{\otimes n} - 1 = X^{\otimes n}(2\left|1\right\rangle^{\otimes n}\left\langle 1\right|^{\otimes n} - 1)X^{\otimes n}$.

$$\begin{aligned} U_3 &= 1 - 2\left|1\right\rangle^{\otimes n}\left\langle 1\right|^{\otimes n} = 1 - \left|1\right\rangle^{\otimes n-1}\left\langle 1\right|^{\otimes n-1}(1_n - Z_n) \\ &= (1 - \left|1\right\rangle^{\otimes n-1}\left\langle 1\right|^{\otimes n-1})1_n + \left|1\right\rangle^{\otimes n-1}\left\langle 1\right|^{\otimes n-1} Z_n \end{aligned} \tag{1}$$

is the N qubit controlled Z gate (I get a minus sign iff all qubits are 1).

5. $Z = HXH$.

$$\begin{aligned} U_3 &= (1 - \left|1\right\rangle^{\otimes n-1}\left\langle 1\right|^{\otimes n-1})H_nH_n + \left|1\right\rangle^{\otimes n-1}\left\langle 1\right|^{\otimes n-1} H_nX_nH_n \\ &= H_n[(1 - \left|1\right\rangle^{\otimes n-1}\left\langle 1\right|^{\otimes n-1})1 + \left|1\right\rangle^{\otimes n-1}\left\langle 1\right|^{\otimes n-1} X]H_n. \end{aligned} \tag{2}$$

The gate in the middle is the $n-$qubit Toffoli gate T_n.

6.

$$U_\psi = H^{\otimes n}U_2H^{\otimes n} = -H^{\otimes n}X^{\otimes n}U_3X^{\otimes n}H^{\otimes n} = -H^{\otimes n}X^{\otimes n}H_nT_nH_nX^{\otimes n}H^{\otimes n} \tag{3}$$