Quantum algorithms 2022/2023: Exercices 3

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1 Implementation of the quantum Fourier transform

Ref: Nielsen and Chuang. The quantum Fourier transform realizes the transformation

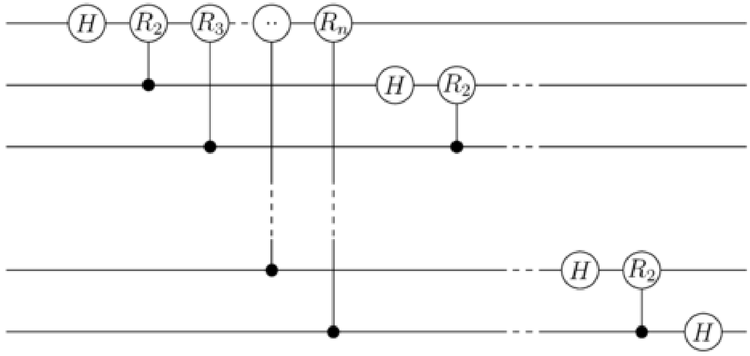
$$U|j\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i jk/N} |k\rangle, \tag{1}$$

with $j, k = 0, \ldots, N-1$. Our goal is to implement this transformation for $N = 2^n$, using a circuit of n qubits.

1. Write any integer j in terms of a binary representation $j = j_1 \ldots j_n$. In our quantum circuit, j_l will represent the state of qubit l.
2. We use the notation $0.j_l \ldots j_n = j_l/2 + \ldots j_n/2^{n-l+1}$. Show that

$$U|j\rangle = \frac{1}{2^{n/2}}(|0\rangle + e^{2i\pi 0.j_n}|1\rangle)(|0\rangle + e^{2i\pi 0.j_{n-1}j_n}|1\rangle)\ldots(|0\rangle + e^{2i\pi 0.j_1\ldots j_n}|1\rangle) \tag{2}$$

3. We show the circuit of the quantum Fourier transform.



   The single qubit gate R_k is defined as

$$R_k = \begin{bmatrix} 1 & 0 \\ 0 & e^{2i\pi/2^k} \end{bmatrix}. \tag{3}$$

   How is a basis $|j\rangle$ transformed after the first controlled R_2 rotation? After the first R_n rotation?

4. Write down the state at the end of the circuit. Conclusion.

2 Factorizing 21 with Shor's algorithm

We take $N = 21$.

1. **Classical part** Assume we randomly pick $a = 2$. Show that the function $f(x) = a^x \bmod(N)$ is 6 periodic.
2. Find two non-trivial divisors of N.
3. **The quantum subroutine** The quantum subroutine of Shor's algorithm consists in finding the period $r = 6$ of $f(x)$. How many qubits do we need to implement this algorithm?
4. Write the state of the system after modular exponentiation.
5. Write the state after inverse quantum Fourier transform and the probability $P(y)$ to observe the bitstring y after measuring the first q qubits.
6. Plot the function $P(y)$ and extract the three most likely measured bitstrings.
7. The continued fraction algorithm is a classical algorithm that gives the closest fraction p/r from the measured y/Q rational, with a maximum $r_{\max}$ value for r. In Python, this is implemented as fractions.Fraction(float).limit_denominator(rmax).

   Give the attributed value for each most likely bitstring r. Comment.

8. Repeat the same exercice, aiming at factorizing 35.