Quantum algorithms: Exercices 4

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1 Measurement of ZZ stabilizers

The error syndromes in the three qubit bit flip code correspond to the measurement of the operators Z_1Z_2 and Z_2Z_3. Let us consider the measurement of Z_1Z_2 in this exercice (We forget for this exercice about the existence of the third physical qubit).

1. Write Z_1Z_2 as a function of two projection operators P_1, P_{-1}.
2. Given a state $|\psi\rangle$ representing the two qubit state. We perform a measurement of the operator Z_1Z_2. What are the possible measurements outcomes and projected states ?
3. Let us use an ancilla qubit to physically implement this measurement process, using the circuit shown in Lecture 4. Write the state of the system before and after the measurement of the ancilla qubit, and show that, effectively, the ancilla realizes the measurement of Z_1Z_2. It will be convenient to use the identity $|\psi\rangle = (|00\rangle\langle 00| + |11\rangle\langle 11| + |10\rangle\langle 10| + |01\rangle\langle 01|)|\psi\rangle$.

2 The three qubit phase flip code

The three qubit phase flip code can correct against qubit phase errors Z_1, Z_2, Z_3. This is achieved via the error syndromes associated with the measurements of the operators X_1X_2 and X_2X_3.

1. Define the two logical states $|0_L\rangle$, $|1_L\rangle$
2. Show that the code can correct against phase errors.
3. Write a circuit to encode a logical qubit from a physical qubit $|\psi\rangle = a|0\rangle + b|1\rangle$.
4. Write a circuit to perform an error syndrome X_1X_2 using an ancilla qubit.

3 Fault tolerance with the surface code (from arXiv:1208.0928)

We consider the surface code. We will illustrate the concept of fault tolerance by studying the scaling of false detection of X errors with increasing sizes of the code.

1. Consider a single row of the code of length $d = 5$ (number of white physical qubits)

   Describe the state of the system after initialization, and after one bit flip error X_i ($i \in \{1, 5\}$).

2. Suppose the error syndrome step gives $-1, 1, -1, 1$. Give a possible error assignment with two errors. Give a possible error assignment with three errors.
3. Assume a physical qubit error occurs with probability p after one 'circuit operation'. A logical operation is considered to involve in average 8 circuit operations. What is the probability of a physical qubit error qubit after one logical operation?
4. For an arbitrary value of odd d, the most likely undetected errors corresponds to an error of $d_e = (d+1)/2$ qubits (wrongly attributed to the complementary error assignement with $d - (d+1)/2 = (d-1)/2$). What is the probability p_L for such a logical error as a function of p and d, after one logical operation?
5. Adapt the expression of the logical error probability for a $2D$ surface code. Plot p_L as a function of p and d. Comment this result in relation with the notion of fault tolerance.