Quantum Algorithms: Exercices 4

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1 Measurement of ZZ stabilizers

The error syndromes in the three qubit bit flip code correspond to the measurement of the operators Z_1Z_2 and Z_2Z_3. Let us consider the measurement of Z_1Z_2 in this exercice.

1. An error syndrome associated with Z_1Z_2 consists in projecting the state following Born's rules. We define P_1, P_{-1} the projection operators associated with the eigenvalue $\epsilon = \pm 1$ of the operator Z_1Z_2 ($P_1 + P_{-1} = 1$). Then after the measurement, the state is transformed as

$$|\psi\rangle \to P_\epsilon |\psi\rangle \tag{1}$$

   with probability $\langle\psi| P_\epsilon |\psi\rangle$.

2. We first entangle the ancilla qubit with the two physical qubits via two CNOTs. We obtain

$$\begin{aligned} |\psi\rangle|0\rangle &= (|00\rangle\langle 00| + |11\rangle\langle 11| + |00\rangle\langle 00| + |11\rangle\langle 11|)|\psi\rangle)|0\rangle \\ |\psi\rangle|0\rangle &\to (|00\rangle\langle 00| + |11\rangle\langle 11|)|\psi\rangle|0\rangle + (|01\rangle\langle 01| + |10\rangle\langle 10|)|\psi\rangle|1\rangle \\ &= |\psi'\rangle = P_1|\psi\rangle|0\rangle + P_{-1}|\psi\rangle|1\rangle \end{aligned} \tag{2}$$

Therefore, a measurement of the ancilla qubit in the 0 state occurs with probability $\langle\psi'|\,|0\rangle\langle 0|\,|\psi'\rangle = \langle\psi| P_1 |\psi\rangle$, and projects the state in $P_1|\psi\rangle|0\rangle$. Same thing for P_{-1}. Thus, an ancilla qubit allows us to realize the mesurement of Z_1Z_2 as described above.

2 The three qubit phase flip code

The three qubit phase flip code can correct against qubit phase errors Z_1, Z_2, Z_3 based on error syndromes associated with the measurement of X_1X_2, X_2X_3, X_1X_3.

1. We define the error set $E = \{I, Z_1, Z_2, Z_3\}$. For each error, we obtain a unique error syndrome.

2. In the spirit of the bit flip code, we define

$$\begin{aligned} |0_L\rangle &= HHH|000\rangle = (|0\rangle + |1\rangle)(|0\rangle + |1\rangle)(|0\rangle + |1\rangle) \\ |1_L\rangle &= HHH|111\rangle = (|0\rangle - |1\rangle)(|0\rangle - |1\rangle)(|0\rangle - |1\rangle) \end{aligned} \tag{3}$$

   The two states are orthonal and stabilized by S, i.e $X_1X_2|a_L\rangle = |a_L\rangle$ (using $XH|0\rangle = X(|0\rangle + |1\rangle) = (|0\rangle + |1\rangle) = H|0\rangle$, and $XH|1\rangle = -H|1\rangle$).

   Suppose a phase error Z_1 occurs on the first qubit

$$a|0_L\rangle + b|1_L\rangle \to a(|0\rangle - |1\rangle)(|0\rangle + |1\rangle)(|0\rangle + |1\rangle) + b(|0\rangle + |1\rangle)(|0\rangle - |1\rangle)(|0\rangle - |1\rangle) \tag{4}$$

   Then the error syndromes give $\langle X_1X_2\rangle = -1$, $\langle X_2X_3\rangle = 1$. I detect this error, which I can fix by applying Z_1.

3. First two CNOTS targeted on the second and third qubit

$$(a|0\rangle + b|1\rangle)|0\rangle|0\rangle \to (a|00\rangle + b|11\rangle)|0\rangle \to a|000\rangle + b|111\rangle \tag{5}$$

   Then three Hadamard

$$a|000\rangle + b|111\rangle \to aHHH|000\rangle + bHHH|111\rangle\,. \tag{6}$$

4. $\langle X_1X_2\rangle = \langle H_1H_2Z_1Z_2H_1H_2\rangle$. This means I can repeat the recipe of the previous exercise with application of two Hadamard gates before and after the CNOTs.

   We obtain

$$\begin{aligned} H^{\otimes 3}|\psi\rangle &= (|0\rangle_1\langle 0| + |1\rangle_1\langle 1|)(|0\rangle_2\langle 0| + |1\rangle_2\langle 1|)H^{\otimes 3}|\psi\rangle \\ |\psi\rangle|0\rangle &\to H^{\otimes 3}(|00\rangle\langle 00| + |11\rangle\langle 11|)H^{\otimes 3}|\psi\rangle|0\rangle + H^{\otimes 3}(|01\rangle\langle 01| + |10\rangle\langle 10|)H^{\otimes 3}|\psi\rangle|1\rangle \\ &= |\psi'\rangle = P_X(1)|\psi\rangle|0\rangle + P_X(-1)|\psi\rangle|1\rangle \end{aligned} \tag{7}$$

3 Fault tolerance with the surface code

Adapted from https://arxiv.org/pdf/1208.0928.pdf.

1. Consider a single row of the code of length $d = 5$ (number of white physical qubits).

By definition of a stabilizer code, any logical state $|\psi\rangle$, i.e the code world is stabilized by the stabilizers, i.e for any $i = 1, \ldots, 4$

$$Z_i Z_{i+1} |\psi\rangle = |\psi\rangle . \tag{8}$$

For an X_i error on a certain qubit, the state becomes $|\psi'\rangle = X_i |\psi\rangle$. This will be detected via the measurements of

$$\langle\psi'| Z_i Z_{i\pm1} |\psi'\rangle = \langle\psi| X_i Z_i X_i Z_{i\pm1} |\psi\rangle = -\langle\psi| Z_i Z_{i\pm1} |\psi\rangle = -1. \tag{9}$$

2. Suppose the error syndrome step gives $-1, 1, -1, 1$. With two errors, the assignement is $X_2 X_3$. The complementary error $X_1 X_4 X_5$ would give the same syndrome, giving rise to a logical X error.

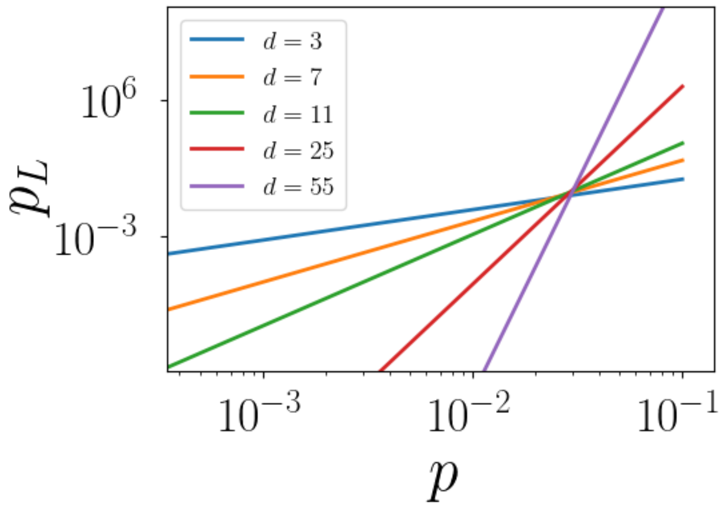
3. A qubit X error occurs with probability p during 8 steps of a logical operation. The probability of an error is therefore $1 - (1-p)^8 \approx 8p$.

4. A given pattern of such error occurs with probability $(8p)^{d_e}(1-8p)^{d-d_e} \approx (8p)^{d_e}$. There are $C_{d,de} = d!/[(d-d_e)!d_e!]$ such patterns. This gives a logical error

$$p_L = C_{d,(d+1)/2}(8p)^{(d+1)/2} \tag{10}$$

5. In a $2D$ code, the logical error rate is approximately multiplied by d, because:

$$p_L^{(2D)} = (1-p_L)^d \approx p_L d = dC_{d,(d+1)/2}(8p)^{(d+1)/2} \tag{11}$$



For $p < p_c \approx 0.03$, the logical error rate decreases as d increases, i.e we can reach arbitrary accuracy by adding physical qubits. This is the notion of fault tolerance.