Quantum Algorithms: Exercices 6

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1 Density matrix

The density matrix ρ summarizes all the physical properties of a quantum S, embedded in an environment E (For instance , think of a quantum computer S subject to a decoherence environment),

1. To introduce ρ, we first write the expectation value of an arbitrary operator $O = O_S \otimes 1$ acting on S, for a general wavefunction
$$|\psi\rangle = \sum_{m,n} c_{m,n} |m\rangle_S \otimes |n\rangle_E \tag{1}$$
describing the combined state of the system and environnement. Here, $\{|m\rangle_S\}$, $\{|n\rangle_E\}$ are orthonormal bases descring the system, environment respectively. Write this expectation value of O as a function of $\{c_{m,n}\}$.

2. We introduce the partial trace operation as
$$\mathrm{tr}_E(A) = \sum_n {}_E\langle n|A|n\rangle_E \tag{2}$$
The linear operation ${}_E\langle n| \cdot$ is defined as ${}_E\langle n'| (|m\rangle_S \otimes |n\rangle_E) = \delta_{n,n'} |m\rangle_S$. We define the density matrix of the system S as $\rho = \mathrm{tr}_E(|\psi\rangle\langle\psi|)$. Write ρ as a function of the decomposition $\{c_{m,n}\}$.

3. Show that $\langle\psi|O|\psi\rangle = \mathrm{Tr}(\rho O_S)$. Comment

4. Show that ρ is Hermitian positive semi-definite, and that it has unit trace

5. Write how the density matrix evolves via a unitary operation U acting on the system S?

2 Pure states and decoherence

6. We introduce $p_2 = \mathrm{tr}(\rho^2)$. Show that a 'pure state' $\rho = |\phi\rangle_S \langle\phi|$ associated with a decoupled environment $|\psi\rangle = |\phi\rangle_S |\phi\rangle_E$ has $p_2 = 1$. p_2 is called the purity and measures to which extent a system is decoupled from its environment.

7. Show that the purity is constant under unitary operation.

8. We introduce the depolarization channel $\mathcal{L}$ of rate p, for a system of q qubits
$$\rho' = \mathcal{L}(\rho) = (1-p)\rho + (p/2^q)\mathbf{1} \tag{3}$$
Calculate the purity of ρ', and discuss the extreme cases $p = 0, 1$ for ρ a pure state

3 Quantum state tomography

9. Quantum state tomography describes a protocol to measure the density matrix ρ in a quantum computer with q qubits. It is based on decomposing ρ is a basis of 'Pauli strings' with $\sigma = \bigotimes_{i=1}^q \sigma_i$, $\sigma_i = 1_i, X_i, Y_i, Z_i$.
$$\rho = \sum_\sigma c_\sigma \sigma \tag{4}$$
Show that $\mathrm{Tr}(\sigma\sigma') = \delta_{\sigma,\sigma'}$ Write the expression of c_σ as a function of ρ and σ.

10. Write the probability to observe a given bitstring $s = s_1, \ldots, s_q$ as a function of the density matrix.

11. Using the identities $X = HZH$, $Y = SXS^\dagger = SHZHS^\dagger$, show that we can write a Pauli operator as $\sigma_i = U_i^\dagger(|0\rangle\langle 0| + \epsilon_i |1\rangle\langle 1|)U_i$.

12. Write a quantum circuit to measure each term c_σ, i.e to perform quantum state tomography.