Quantum Algorithms: Exercices 6

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1 Density matrix

The density matrix ρ summarizes all the physical properties of a quantum S, embedded in an environment E (For instance , think of a quantum computer S subject to a decoherence environment),

1. To introduce ρ, we first write the expectation value of an arbitrary operator $O = O_S \otimes 1$ acting on S, for a general wavefunction

$$|\psi\rangle = \sum_{m,n} c_{m,n} |m\rangle_S \otimes |n\rangle_E \tag{1}$$

describing the combined state of the system and environnement. Here, $\{|m\rangle_S\}$, $\{|n\rangle_E\}$ are orthonormal bases descring the system, environment respectively. Write this expectation value of O as a function of $\{c_{m,n}\}$.

Solution:

$$\langle\psi|O|\psi\rangle = \sum_{m,n,m'} c^*_{m,n} c_{m',n} \langle m|O_S|m'\rangle \tag{2}$$

2. We introduce the partial trace operation as

$$\mathrm{tr}_E(A) = \sum_n {}_E\langle n|A|n\rangle_E \tag{3}$$

The linear operation ${}_E\langle n| \cdot$ is defined as ${}_E\langle n'| (|m\rangle_S \otimes |n\rangle_E) = \delta_{n,n'} |m\rangle_S$. We define the density matrix of the system S as $\rho = \mathrm{tr}_E(|\psi\rangle\langle\psi|)$. Write ρ as a function of the decomposition $\{c_{m,n}\}$.

Solution:

$$\rho = \sum_{m,m',n,n',n''} c^*_{m,n} c_{m',n'}\, {}_E\langle n''| \, |m'\rangle_S \otimes |n'\rangle_E \langle m|_S \otimes \langle n|_E \, |n''\rangle_E = \sum_{m,m',n} c^*_{m,n} c_{m',n} |m'\rangle_S \otimes \langle m|_S \tag{4}$$

3. Show that $\langle\psi|O|\psi\rangle = \mathrm{Tr}(\rho O_S)$. Comment

Solution:

$$\mathrm{Tr}(\rho O_S) = \sum_{m,n,m'} c^*_{m,n} c_{m',n} \langle m|O_S|m'\rangle = \langle\psi|O|\psi\rangle \tag{5}$$

The expression of ρ allows us to extract the expectation value of any observable of the system.

4. Show that ρ is Hermitian positive semi-definite, and that it has unit trace

Solution:

$$\rho^\dagger = \sum_{m,m',n} c_{m,n} c^*_{m',n} |m\rangle_S \otimes \langle m'|_S = \rho \tag{6}$$

For a given $|\phi\rangle = \sum_l \alpha_l |l\rangle$

$$\langle\phi|\rho|\phi\rangle = \sum_{l,l',n} (\alpha^*_l c_{l,n})(\alpha_{l'} c^*_{l',n}) \geq 0 \tag{7}$$

$$\mathrm{Tr}(\rho) = \sum_{m,n} c^*_{m,n} c_{m,n} = 1 \tag{8}$$

5. Write how the density matrix evolves via a unitary operation U acting on the system S? **Solution:**

$$|\psi'\rangle = (U \otimes 1)|\psi\rangle \tag{9}$$

$$\rho' = \mathrm{Tr}_E[(U \otimes 1)|\psi\rangle\langle\psi|(U^\dagger \otimes 1)] = U\rho U^\dagger \tag{10}$$

2 Pure states and decoherence

6. We introduce $p_2 = \text{tr}(\rho^2)$. Show that a 'pure state' $\rho = |\phi\rangle_S \langle\phi|$ associated with a decoupled environment $|\psi\rangle = |\phi\rangle_S |\phi\rangle_E$ has $p_2 = 1$. p_2 is called the purity and measures to which extent a system is decoupled from its environment.

   Solution:
$$p_2 = \text{Tr}(|\phi\rangle\langle\phi| |\phi\rangle\langle\phi|) = \text{tr}(|\phi\rangle\langle\phi|) = 1 \tag{11}$$

7. Show that the purity is constant under unitary operation.

   Solution:
$$p_2(\rho') = \text{Tr}(U\rho U^\dagger U\rho U^\dagger) = p_2 \tag{12}$$

8. We introduce the depolarization channel $\mathcal{L}$ of rate p, for a system of q qubits
$$\rho' = \mathcal{L}(\rho) = (1-p)\rho + (p/2^q)\mathbf{1} \tag{13}$$
   Calculate the purity of ρ', and discuss the extreme cases $p = 0, 1$ for ρ a pure state

   Solution:
$$p_2(\rho') = (1-p)^2 p_2 + 2p(1-p)/2^q + p^2/2^q = (1-p)^2 p_2 + p(2-p)/2^q \tag{14}$$
   For ρ a pure state, we obtain $p_2(\rho') = p_2 = 1$ for $p = 0$, and $p_2(\rho') = 1/2^q$. Note that the state $\mathbf{1}/2^q$ with purity $1/2^q$ is called the maximally mixed state.

3 Quantum state tomography

9. Quantum state tomography describes a protocol to measure the density matrix ρ in a quantum computer with q qubits. It is based on decomposing ρ is a basis of 'Pauli strings' with $\sigma = \bigotimes_{i=1}^q \sigma_i$, $\sigma_i = 1_i, X_i, Y_i, Z_i$.
$$\rho = \sum_\sigma c_\sigma \sigma \tag{15}$$
   Show that $\text{Tr}(\sigma\sigma') = \delta_{\sigma,\sigma'}$ Write the expression of c_σ as a function of ρ and σ.

   Solution:
$$\text{Tr}(\sigma\sigma') = \text{Tr}(\bigotimes_i \sigma_i\sigma_i') = \prod_i \text{Tr}(\sigma_i\sigma_i') = \prod_i \delta_{\sigma_i,\sigma_i'} = \delta_{\sigma,\sigma'} \tag{16}$$
$$\text{Tr}(\rho\sigma) = \sum_{\sigma'} \text{Tr}(c_\sigma' \sigma\sigma') = c_\sigma 2^q \tag{17}$$

10. Write the probability to observe a given bitstring $s = s_1, \dots, s_q$ as a function of the density matrix.

    Solution: We have $P(s)$ the expectation value of the operator $O = |s\rangle\langle s|$. So
$$P(s) = \text{Tr}(\rho |s\rangle\langle s|) = \langle s|\rho|s\rangle \tag{18}$$

11. Using the identities $X = HZH$, $Y = SXS^\dagger = SHZHS^\dagger$, show that we can write a Pauli operator as $\sigma_i = U_i^\dagger(|0\rangle\langle 0| + \epsilon_i |1\rangle\langle 1|)U_i$.

    Solution: For $\sigma_i = 1$, we take $U_i = 1$, and $\epsilon_i = 1$. For $\sigma_i = Z$, we take $U_i = 1$, and $\epsilon_i = -1$. For $\sigma_i = X$, we use $U_i = H$, $\epsilon_i = -1, \dots$

12. Write a quantum circuit to measure each term c_σ, i.e to perform quantum state tomography.

    Solution: We can write
$$\begin{aligned}\sigma &= \sigma_1 \otimes \dots \otimes \sigma_q = U_1^\dagger(|0\rangle\langle 0| + \epsilon_1 |1\rangle\langle 1|)U_1 \dots U_q\dagger(|0\rangle\langle 0| + \epsilon_q |1\rangle\langle 1|))U_q \\ &= U^\dagger(\sum_s f_s |s\rangle\langle s|)U \end{aligned}\tag{19}$$
    with $U = U_1 \otimes \dots U_q$ and f_s is a combination of the ϵ_i numbers, Therefore
$$c_\sigma = 2^{-q}\text{Tr}(\rho U^\dagger \sum_s f_s |s\rangle\langle s| U) = \sum_s f_s P_U(s) \tag{20}$$
    with $P_U(s) = \langle s|\rho_U|s\rangle$, $\rho_U = U\rho U^\dagger$ the state after application of the unitary U. This mean we can perform tomography by applying U on ρ, measuring the Born probabilities $P_U(s)$ and extracting c_σ via the above equations.